University at Albany, SUNY

College of Engineering and Applied Sciences, Computer Science

ISEN/ISCI-210: Discrete Structures

Fall 2018

Homework Set 2

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Assigned Date: Sep 28, 2018 (Friday).
Due Date: Oct 8, 2018 (Monday).
Collaboration Policy. Homeworks will be done individually: each student must hand in their own answers. Use of partial or entire solutions obtained from others or online is strictly prohibited.
Late Policy. No late submissions will be allowed without consent from the instructor. If urgent or unusual circumstances prohibit you from submitting a homework assignment in time, please e-mail me explaining the situation.
Submission Format. Electronic submission of a PDF file is mandatory.

Problem 1: Set (20 points) Let $A = \{a, b, c\}$, $B = \{x, y\}$, $C = \{x \in \mathbf{Z} | 0 \leq x \leq 1\}$, $D = \{x \in \mathbf{N} | x \leq 5\}$, and $E_i = \{1, 2, 3, \ldots, i\}$, find

(a) $A \times B \times C$
(b) $B \times A \times C$
(c) A^2.
(d) C^3
(e) $C \cup D$
(f) $C \cap D$
(g) $C - D$
(h) $D - C$
(i) $\bigcup_{i=1}^{n} E_i$
(j) $\bigcap_{i=1}^{n} E_i$

Problem 2: Function with One-to-one Correspondence (10 points) Determine whether each of these functions is a bijection from $\mathbf{R}$ to $\mathbf{R}$. If it is, then derive its inverse function.

(a) $f(x) = 2x + 1$

(b) $f(x) = x^2 + 1$
(c) $f(x) = x^3$
(d) $f(x) = (x^2 + 1)/(x^2 + 2)$

Problem 3: Function Composition (15 points).

(a) Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\mathbf{R}$ to $\mathbf{R}$.

(b) Find $f + g$ and fg for the functions f and g given (a).

(c) Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $f \circ g = g \circ f$.

Problem 4: Sequence (15 points) Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$

(a) Find a_0, a_1, a_2, a_3, and a_4.

(b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.

(c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.

Problem 5: Sequences and Summation (24 points) Compute each of these sums

(a) $\sum_{i=1}^{8} 3 \times 2^i$

(b) $\sum_{j=1}^{24} (-6j + 5)$

(c) $\sum_{k=1}^{519} \frac{1}{k(k+2)}$

(d) $\sum_{j=0}^{18} (j + (-1)^j)$

(e) $\sum_{j=0}^{18} (2 \cdot 3^j + 3 \cdot 2^j)$

(f) $\sum_{k=100}^{200} (2k + 3)$

(g) $\sum_{i=1}^{3} \sum_{j=1}^{2} (i - j)$

(h) $\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3$

Problem 6: Cardinality of Sets (6 points) Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

(a) the integers greater than 10.
(b) the odd negative integers.
(c) the integers with absolute value less than 1,000,000.
(d) the real numbers between 0 and 2.
(e) the set $A \times \mathbf{Z}^+$ where $A = 2, 3$.
(f) the integers that are multiples of 10.

Problem 7: Matrices (10 points)

(a) Find $\mathbf{A} + \mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$$

(b) Find $\mathbf{AB}$ with $\mathbf{A}$ and $\mathbf{B}$ given in (a).

(c) Find the Boolean product of $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

(d) Find $\mathbf{A}^{[2]}$, $\mathbf{A}^{[3]}$ and $\mathbf{A} \vee \mathbf{A}^{[2]} \vee \mathbf{A}^{[3]}$ with $\mathbf{A}$ given in (c).

[Optional] Extra Points (20 points)

(a) Show that $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$, where $a_0, a_1, \ldots, a_n$ is a sequence of real numbers. This type of sum is called **telescoping**.

(b) Use the identity $1/(k(k+1)) = 1/k - 1/(k+1)$ and (a) to compute $\sum_{k=1}^{n} 1/(k(k+1))$.

(c) Sum both sides of the identity $k^2 - (k-1)^2 = 2k - 1$ from $k = 1$ to $k = n$ and use (a) to find [1] a formula for $\sum_{k=1}^{n} (2k-1)$ (the sum of the first n odd natural numbers), and [2] a formula for $\sum_{k=1}^{n} k$.

(d) Use the technique given in (a), together with the result of (c)[2], to derive the formula for $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. [Hint: Take $a_k = k^3$ in the telescoping sum in (a).]