UNIVERSITY AT ALBANY

State University of New York

Lecture 25: Recursive Algorithms and Basic Counting Rules

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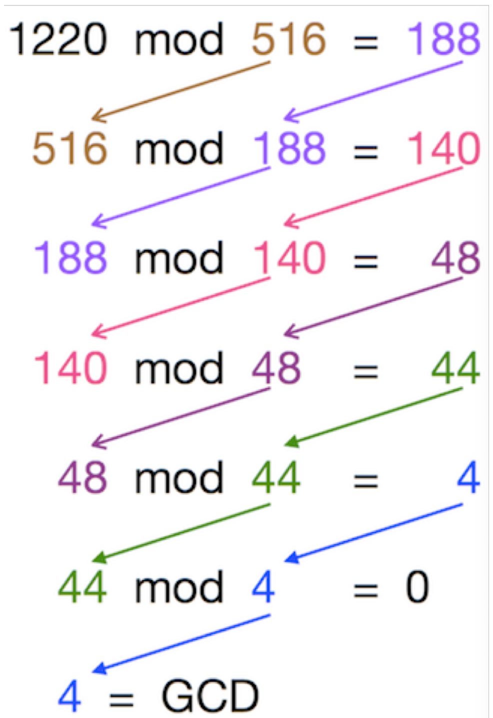
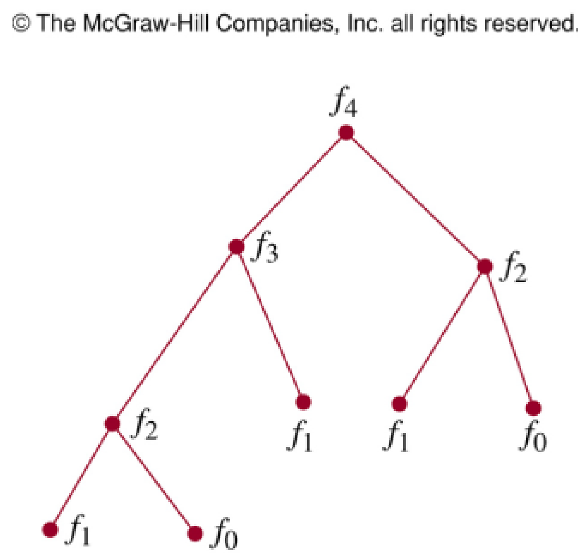
Recap Previous Lecture

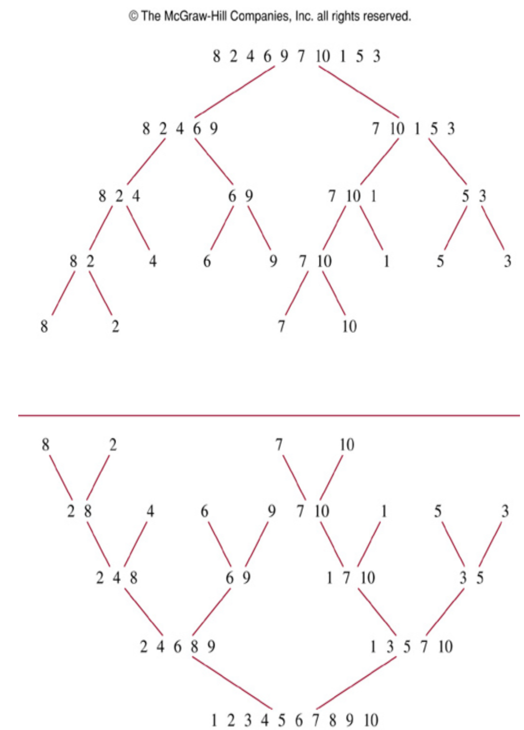
- Recursive Algorithms



GCD



Fibonacci



Merge Sort

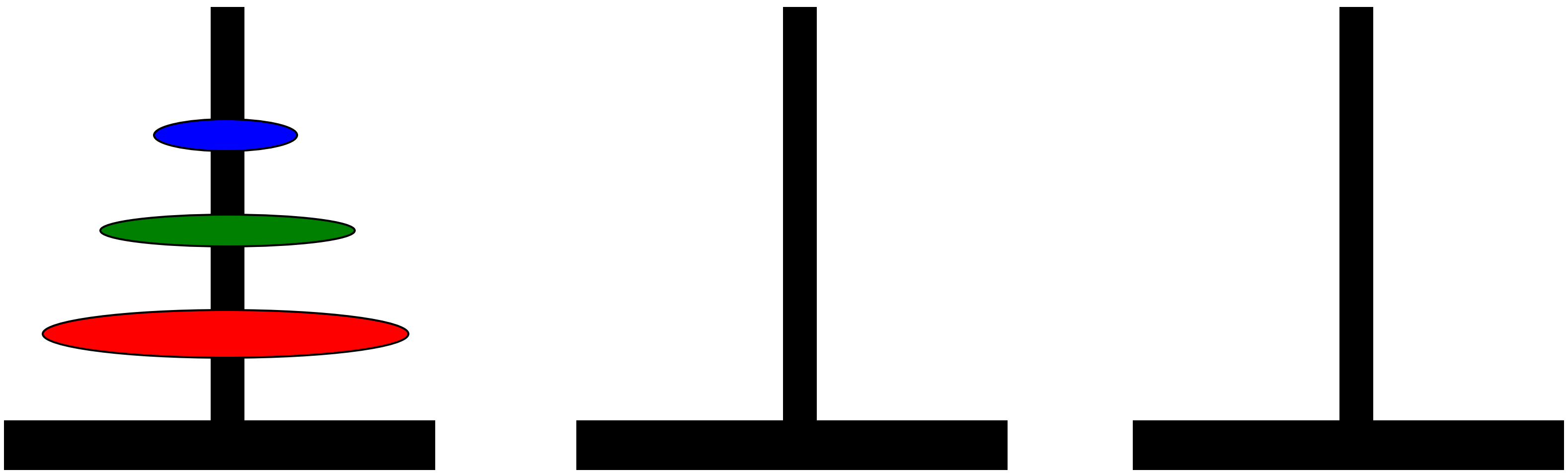
Outline

- Recursive Algorithms: Towers of Hanoi
- Basic Counting Rules
  - Sum Rule
  - Product Rule
  - Generalized Product Rule

Outline

- **Recursive Algorithms: Towers of Hanoi**
- Basic Counting Rules
  - Sum Rule
  - Product Rule
  - Generalized Product Rule

Towers of Hanoi (N=3)



- There are three pegs.
- 3 gold disks, with decreasing sizes, placed on the first peg.
- You need to move all of the disks from the first peg to the second peg.
- Larger disks cannot be placed on top of smaller disks.
- The third peg can be used to temporarily hold disks.

Towers of Hanoi

- The disks must be moved within one week. Assume one disk can be moved in 1 second. Is this possible?
- To create an algorithm to solve this problem, it is convenient to generalize the problem to the “N-disk” problem, where in our case N = 64.

Towers of Hanoi

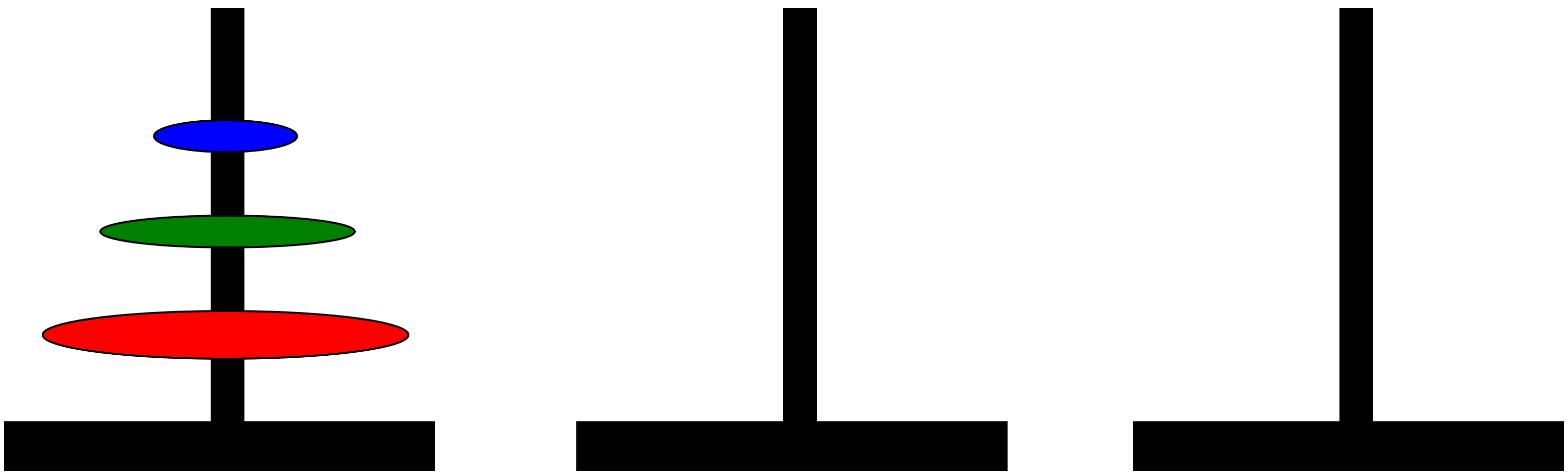
- How to solve it?
- Think recursively!!!!
- Suppose you could solve the problem for n-1 disks, i.e., you can move (n-1) disks from one tower to another, without ever having a large disk on top of a smaller disk. How would you do it for n?

Towers of Hanoi

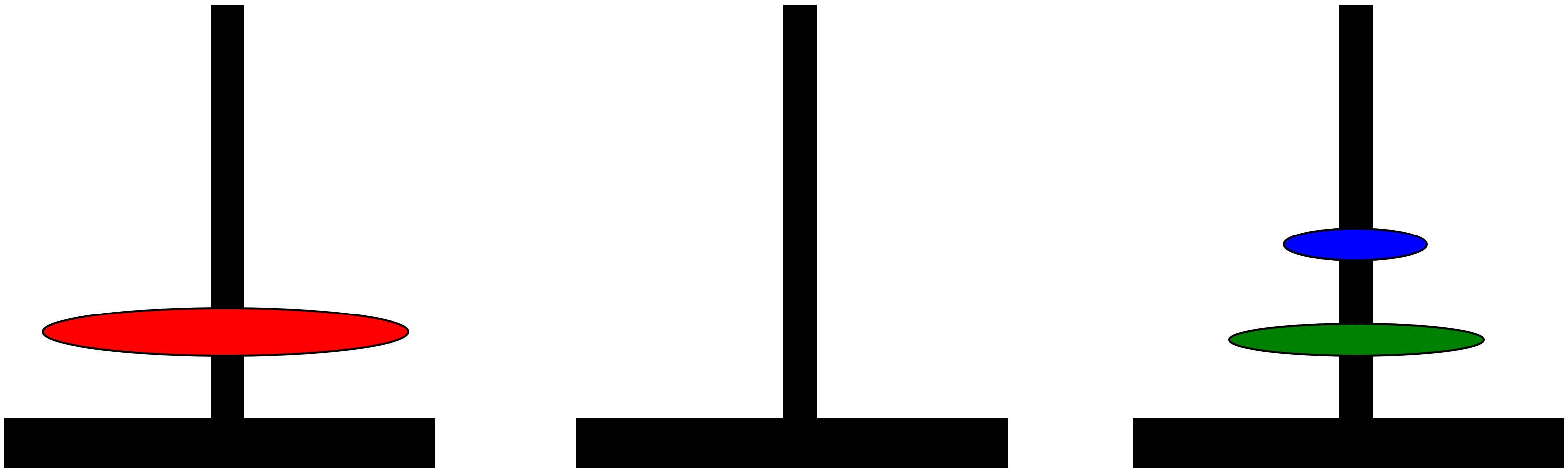
- Solution:

1. Move top (n-1) disks from tower 1 to tower 3 (you can do this by assumption – just pretend the largest ring is not there at all).
2. Move largest ring from tower 1 to tower 2.
3. Move top (n-1) rings from tower 3 to tower 2 (again, you can do this by assumption).

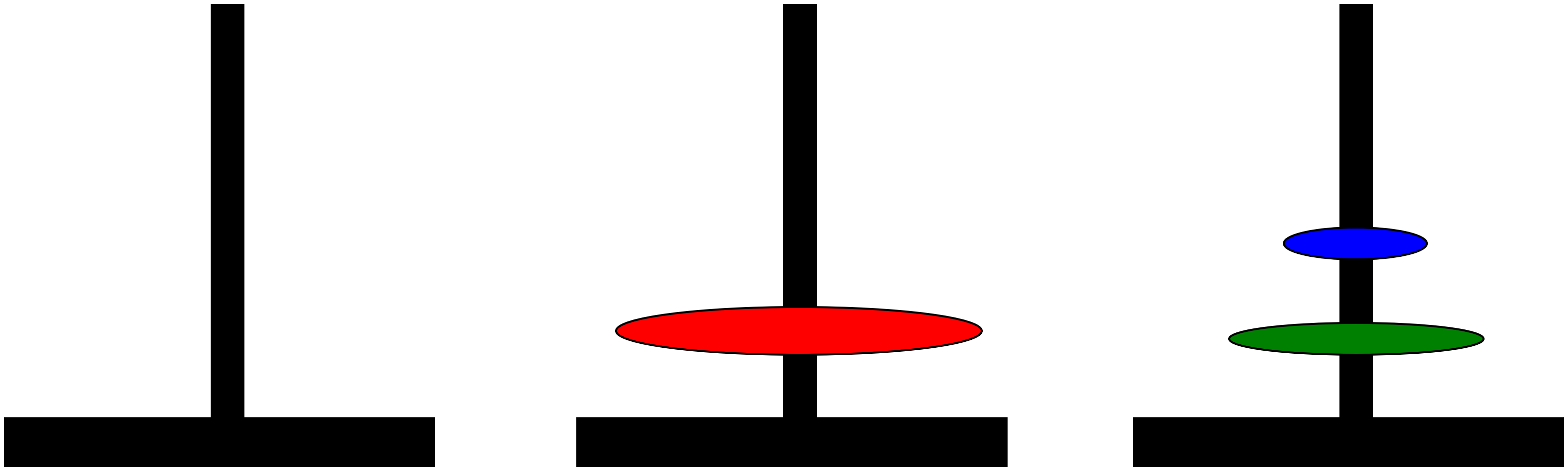
Recursive Solution



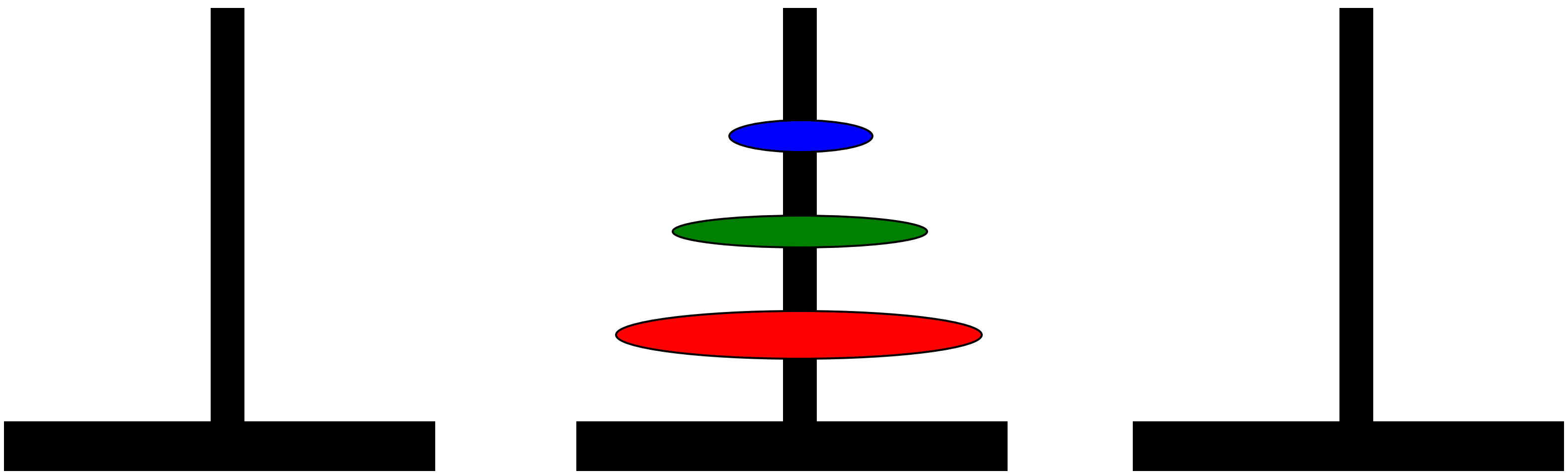
Recursive Solution



Recursive Solution



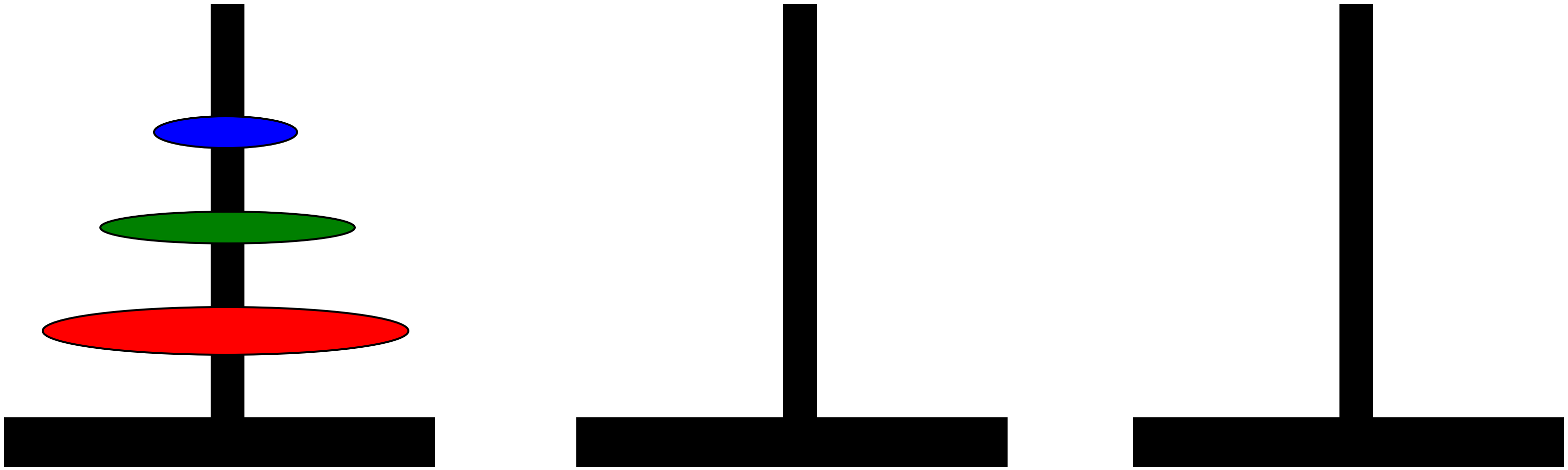
Recursive Solution



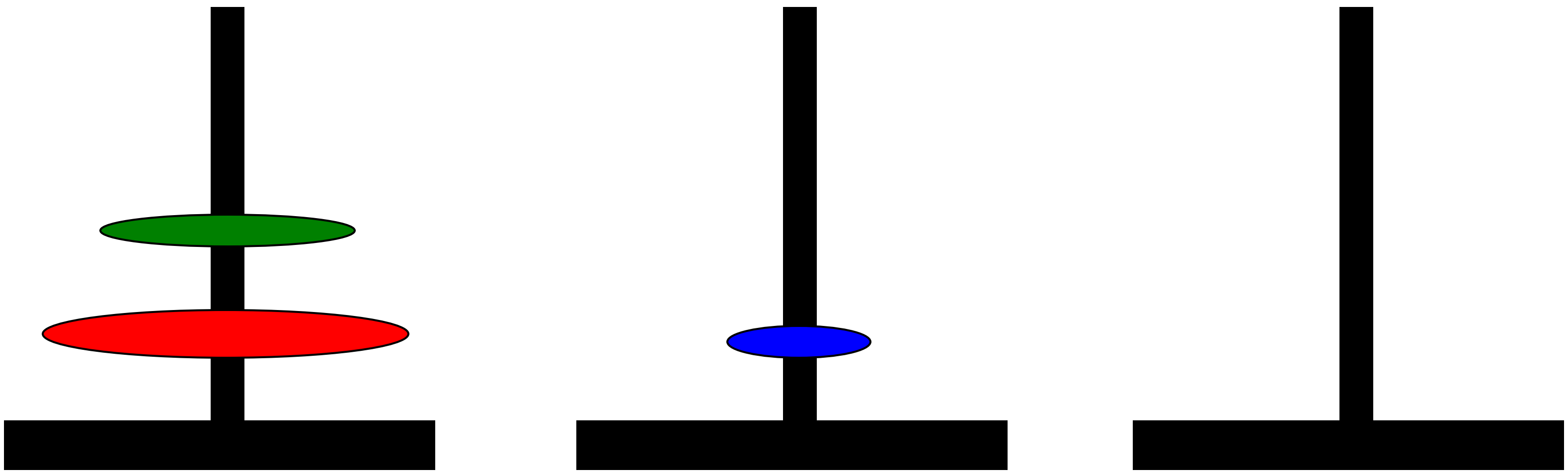
Towers of Hanoi

- **Procedure** *TowerHanoi* (n, a, b, c: integers, 1≤a≤3, 1≤b≤3, 1≤c≤3)
- **if** n= 1 **then**
-       *move*(a,b)
- **else**
- **begin**
-       *TowerHanoi(*n-1, a, c, b)
-       *move*(a,b);
-       *TowerHanoi* (n-1,c,b,a);
- **end**

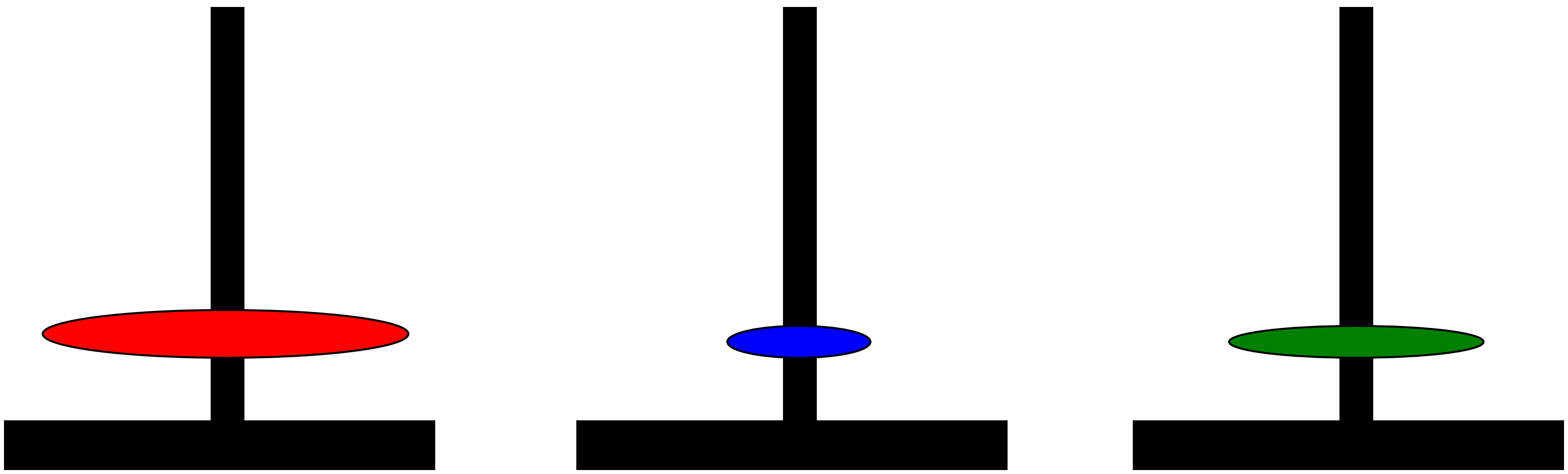
Towers of Hanoi



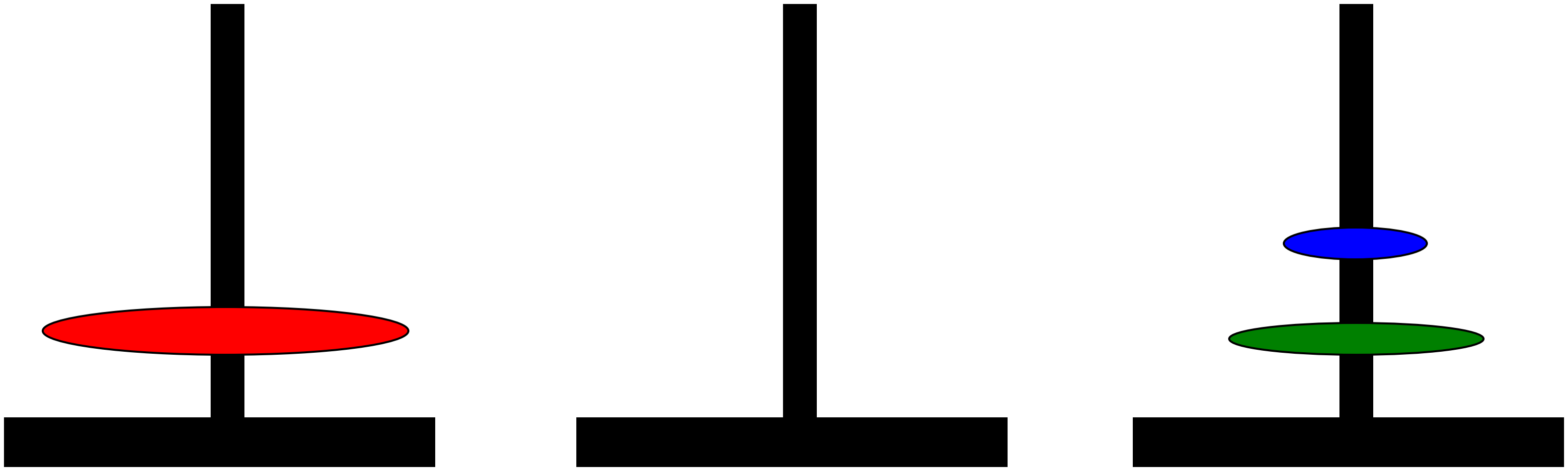
Towers of Hanoi



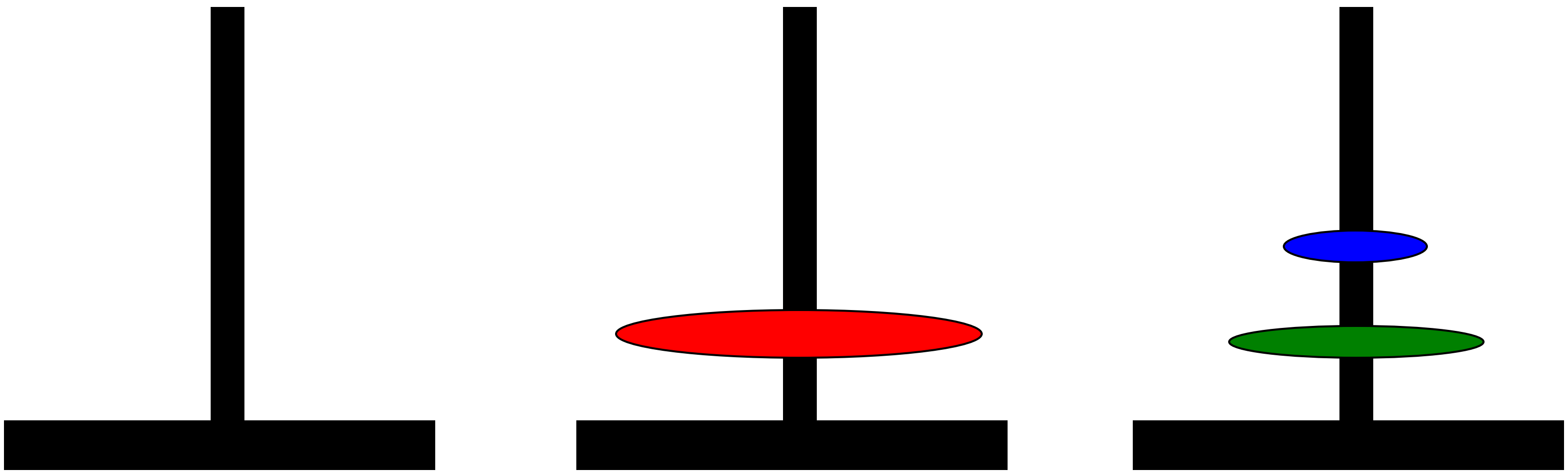
Towers of Hanoi



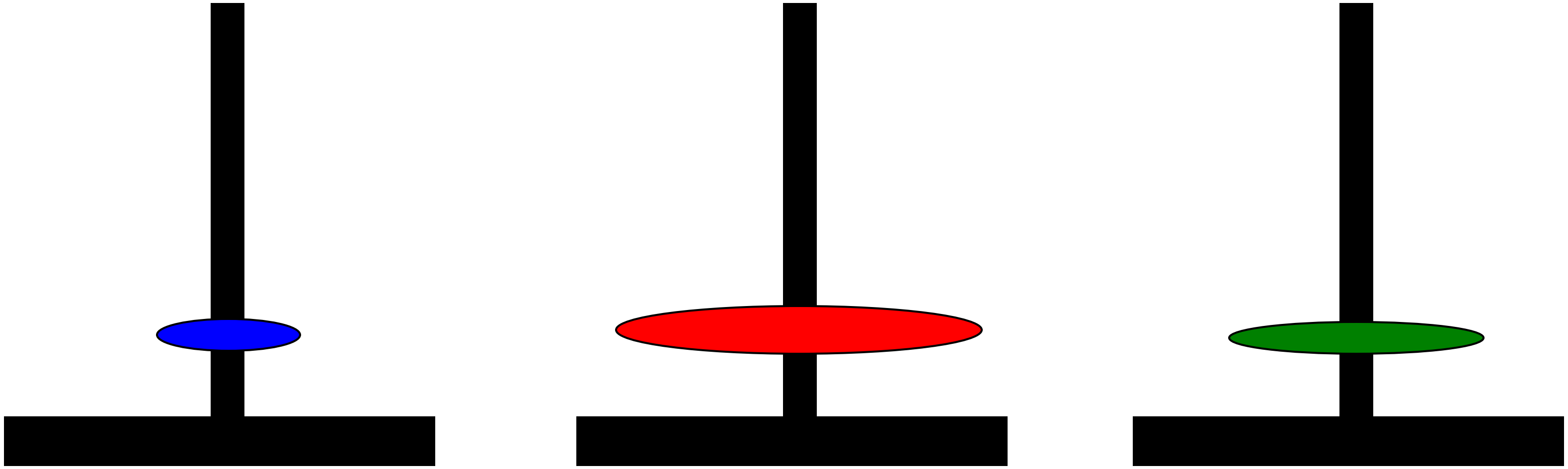
Towers of Hanoi



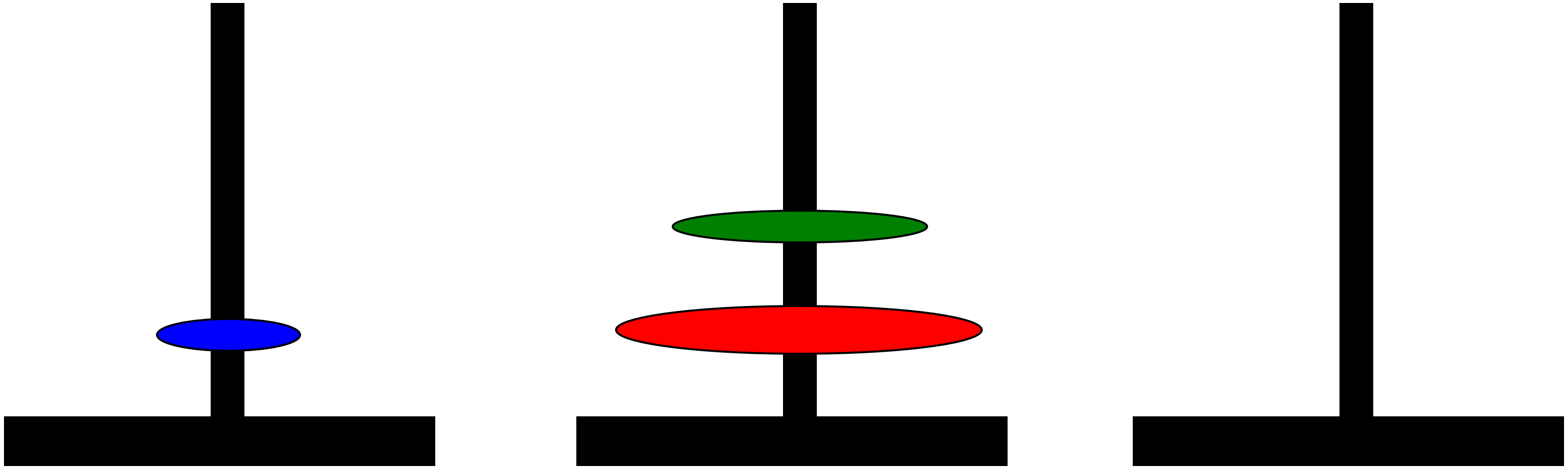
Towers of Hanoi



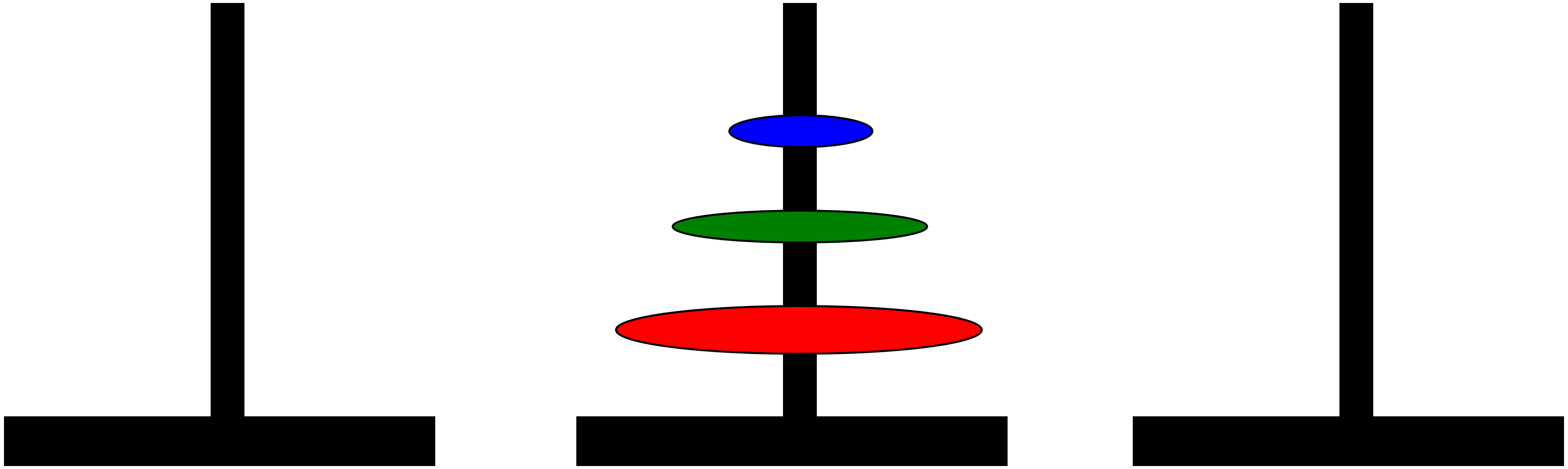
Towers of Hanoi



Towers of Hanoi



Towers of Hanoi



Analysis of Towers of Hanoi

- Hypothesis --- it takes $2^n - 1$ moves to perform TowerHanoi(n,a,b,c) for all positive n.

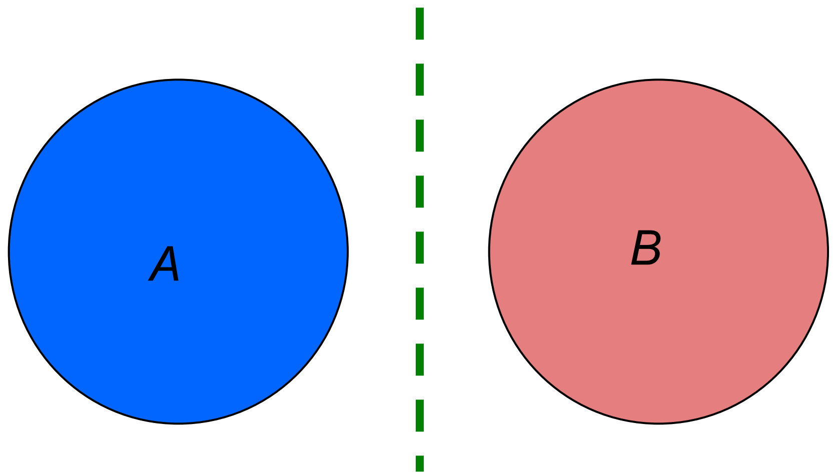
- Proof:
- Basis: P(1) – we can do it using move(a,b) i.e., $2^1 - 1 = 1$
- Inductive Hypothesis: P(n) - it takes $2^n - 1$ moves to perform TowerHanoi(n,a,b,c)
- Inductive Step: In order to perform TowerHanoi(n+1,a,b,c)
- We do: TowerHanoi(n,a,c,b), move(a,c), and TowerHanoi(n,c,b,a);
- Assuming the IH this all takes $2^n - 1 + 1 + 2^n - 1 = 2 \times 2^n - 1 = 2^{(n+1)} - 1$

Outline

- Recursive Algorithms: Towers of Hanoi
- **Basic Counting Rules**
  - Sum Rule
  - Product Rule
  - Generalized Product Rule

Sum Rule

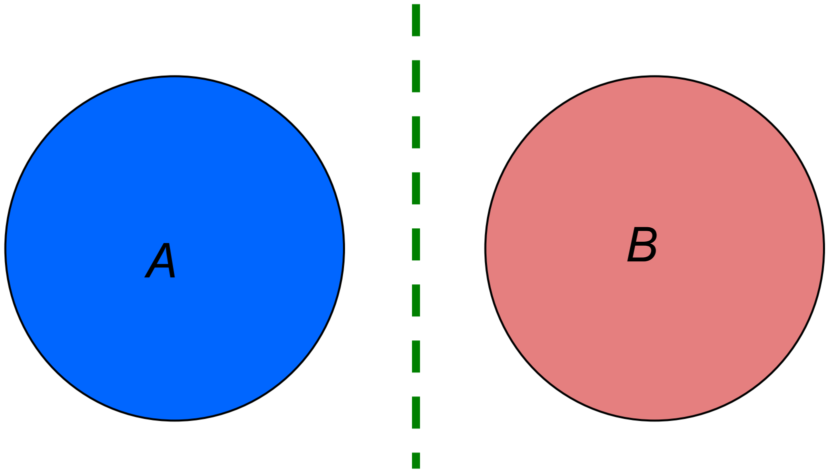
|S|: the number of elements in a set S.



If sets A and B are disjoint, then

$|A \cup B| = |A| + |B|$

Sum Rule



If sets *A* and *B* are disjoint, then

$|A \cup B| = |A| + |B|$

- Class has 43 women, 54 men, so total enrollment = 43 + 54 = 97
- 26 lower case letters, 26 upper case letters, and 10 digits, so total characters = 26+26+10 = 62

Product Rule

Recall that, given two sets A and B, the Cartisean product

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

Fact: If |A| = n and |B| = m, then |AxB| = mn.

A = {a, b, c, d}, B = {1, 2, 3}

$A \times B$ = {(a,1),(a,2),(a,3),
(b,1),(b,2),(b,3),
(c,1),(c,2),(c,3),
(d,1),(d,2),(d,3) }

Example: If there are 4 men and 3 women, there are $4 \times 3 = 12$ possible married couples.

Product Rule

Fact: If |A| = n and |B| = m, then |AxB| = mn.

In general let $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

We can arrange the elements into a table as follows.

$$
\begin{aligned}
A \times B = \{&(a_1,b_1), (a_1,b_2),\ldots, (a_1,b_n),\\
&(a_2,b_1), (a_2,b_2),\ldots, (a_2,b_n),\\
&(a_3,b_1), (a_3,b_2),\ldots, (a_3,b_n),\\
&\qquad\qquad\ldots\\
&(a_m,b_1), (a_m,b_2),\ldots, (a_m,b_n), \}
\end{aligned}
$$

There are m rows, and each row has n elements,

and so there are a total of mn elements.

Product Rule

Fact: $|A_1 x A_2 x \ldots x A_k| = |A_1| x |A_2| x \ldots x |A_k|$.

The formal proof uses mathematical induction.

But the proof idea is not difficult.

We think of $A_1 x A_2 x \ldots x A_k$ as $(\ldots((A_1 x A_2) x A_3) \ldots x A_k)$.

That is, we first construct $A_1 x A_2$, and it is a set of size $|A_1| x |A_2|$.

Then, we construct $(A_1 x A_2) x A_3$, the product of $A_1 x A_2$ and A_3,

and it is a set of size $(|A_1| x |A_2|) x |A_3|$ by the product rule on two sets.

Repeating the argument we can see that $|A_1 x A_2 x \ldots x A_k| = |A_1| x |A_2| x \ldots x |A_k|$.

Example: Counting Strings

What is the number of 10-bit strings?

Let B={0,1}.

The set of 2-bit strings is just BxB.

The set of 10-bit strings is just BxBxBxBxBxBxBxBxBxB, denoted by B^{10}.

By the product rule, |BxB| = |B|x|B| = 2x2 = 4, and

$|B^{10}| = |B|x|B|x|B|x|B|x|B|x|B|x|B|x|B|x|B|x|B| = |B|^{10} = 2^{10} = 1024$.

Example: IP Addresses

What is the number of IP addresses?

An IP address is of the form 192.168.0.123.

There are four numbers, each is between 0 and 255.

Let $B=\{0,1,\ldots,255\}$.

Then the set of IP addresses is just B^4.

By the product rule, $|B^4| = |B|^4 = 256^4 = 4294967296$.

Example: Product Rule

In general we have:

The number of length-n *strings* from an *alphabet* of size m is m^n.

That is, $|B^n| = |B|^n$.

e.g. the number of length-n binary strings is 2^n

the number of length-n strings formed by capital letters is 26^n

Example: Counting Passwords

How many passwords satisfy the following requirements?

- between 6 & 8 characters long
- starts with a letter
- case sensitive
- other characters: digits or letters

First we define the set of letters and the set of digits.

$$L = \{a,b,\ldots,z,A,B,\ldots,Z\}$$

$$D = \{0,1,\ldots,9\}$$

Example: Counting Passwords

L ::= {a,b,…,z,A,B,…,Z}

D ::= {0,1,…,9}

We first count the number of passwords with a specific length.

Let P_n be the set of passwords with length n.

$$P_6 = L\times(L\cup D)\times(L\cup D)\times(L\cup D)\times(L\cup D)\times(L\cup D)$$

$$= L\times(L\cup D)^5$$

$$P_n ::= \text{length } n \text{ passwords}$$

$$= L\times(L\cup D)^{n-1}$$

Example: Counting Passwords

$$\left| L \times (L \cup D)^{n-1} \right| = |L| \cdot |L \cup D|^{n-1}$$ by product rule

$$= |L| \cdot \left(|L| + |D| \right)^{n-1}$$ by sum rule

$$= 52 \cdot 62^{n-1}$$

The set of Passwords:

$$P = P_6 \cup P_7 \cup P_8$$

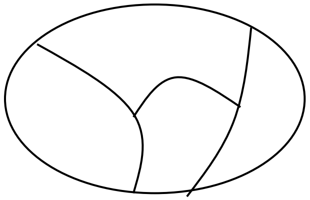
$$|P| = |P_6| + |P_7| + |P_8|$$

$$= 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7$$

$$= 186125210680448$$

$$\approx 19 \cdot 10^{14}$$

counting by partitioning



This is a common technique.

Divide the set into disjoint subsets.

Count each subset and add the answers.

At Least One Seven

How many # 4-digit numbers with at least one 7?

Method 1: count by *1st occurrence* of 7:

7xxx + o7xx + oo7x + ooo7

where x represents any digit from 1 to 10,

while o represent any digit from 1 to 10 except 7.

Clearly, each number containing at least one 7 is in

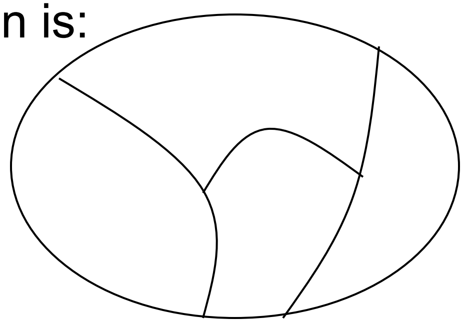
one of the above sets, and these sets are disjoint.

Therefore, the answer to the question is:

$10^3 + 9 \cdot 10^2 + 9^2 \cdot 10 + 9^3 = 3439$

The set of 4-digit numbers with 7 in the first digit.

The set of 4-digit numbers with 7 in the second digit, but the first digit is not 7, and so on.



(counting by partitioning)

At Least One Seven

How many # 4-digit numbers with at least one 7?

Method 2:

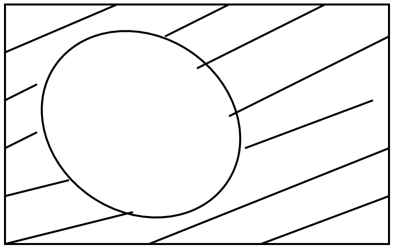
|4-digit numbers with at least one 7|=

|4-digit numbers| – |those with no 7s|

$= 10^4 - 9^4$

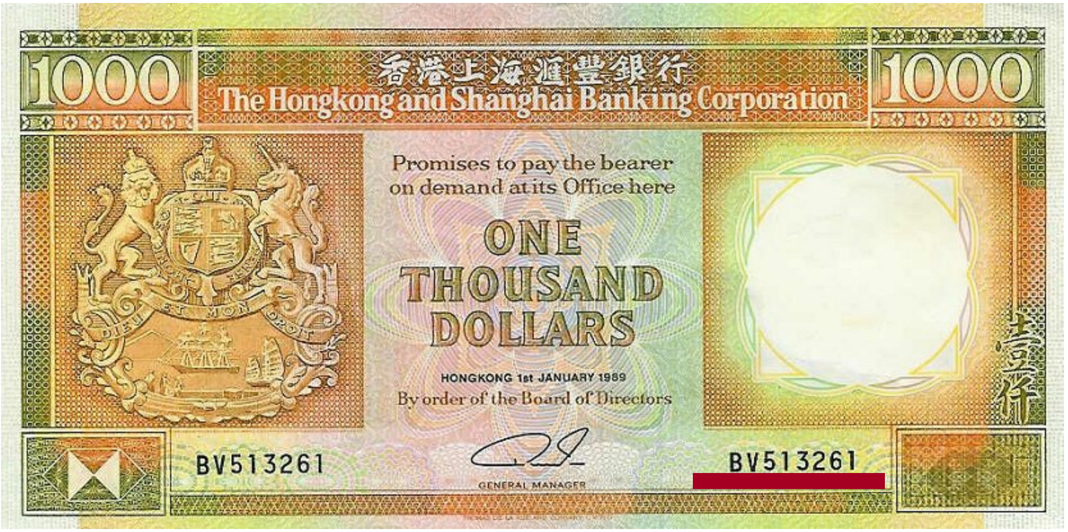
= 3439

(counting the complement)



Counting the complement is a useful technique.

Defective Dollars



A dollar is defective if some digit appears more than once in the 6-digit serial number.

How common are nondefective dollars?

Defective Dollars

How common are nondefective dollars?

10 possible choices for the first digit,

9 possible choices for the second digit, and so on…

So, there are 10x9x8x7x6x5

= 151200 serial number with all its digit different

There are totally 10^6 = 1000000 serial numbers.

So, only about 15% of dollars are nondefective.

Generalized Product Rule

Q a set of length-k sequences. If there are:

n_1 possible 1st elements in sequences,

n_2 possible 2nd elements for each first entry,

n_3 possible 3rd elements for each 1st & 2nd,

…

then, $$|Q| = n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$$

Next class

- Topic: The Pigeonhole Principle and Permutations
- Pre-class reading: Chap 6.2 – 6.2