UNIVERSITY AT ALBANY
State University of New York

Lecture 26: The Pigeonhole Principle, Permutations and Combinations

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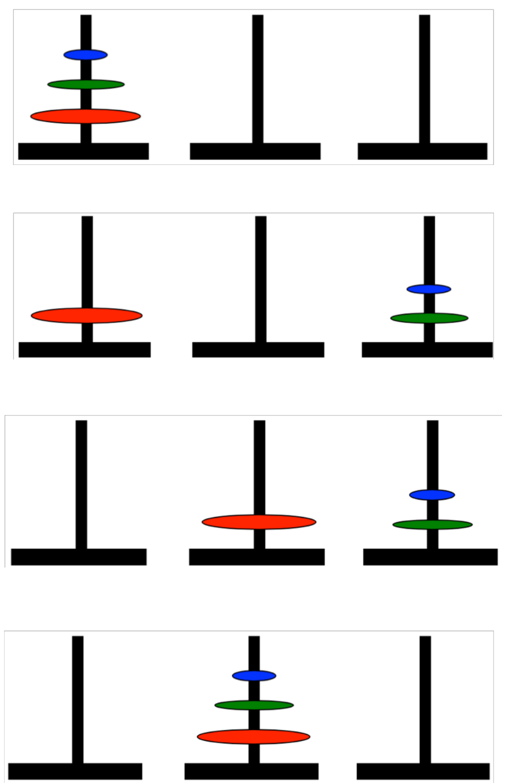
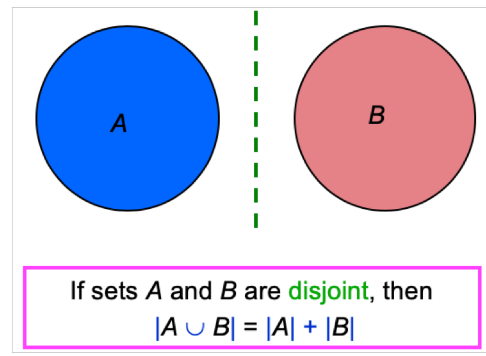
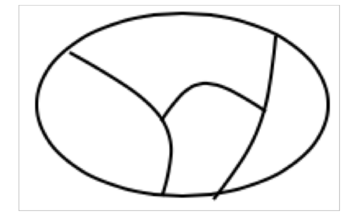
Recap Previous Lecture

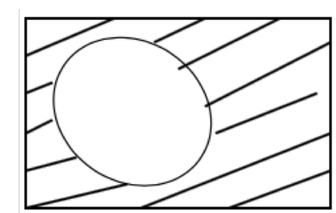
- Recursive Algorithm: Hanoi Tower
- Basic Counting Rules







counting by partitioning



Counting the complement

$$A \times B = \{(a_1,b_1), (a_1,b_2),\ldots, (a_1,b_n), (a_2,b_1), (a_2,b_2),\ldots, (a_2,b_n), (a_3,b_1), (a_3,b_2),\ldots, (a_3,b_n), \ldots (a_m,b_1), (a_m,b_2),\ldots, (a_m,b_n), \}$$

If $|A| = n$ and $|B| = m$, then $|A \times B| = mn$.

Outline

- The Pigeonhole Principle
- Permutations and Combinations

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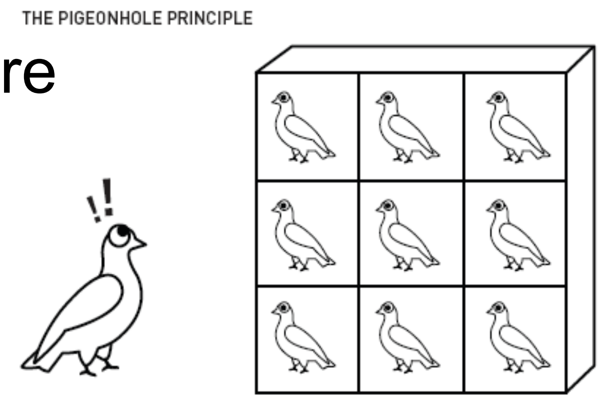
- **The Pigeonhole Principle**
- Permutations and Combinations

Pigeonhole Principle

Motivation:
The mapping of n objects to m buckets
E.g. Hashing.

The principle is used for proofs of certain complexity derivation.

Pigeonhole Principle



Pigeonhole Principle: If n pigeonholes are occupied by n + 1 or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.

Remark: The principle is obvious. No simpler fact or rule to support or prove it.

Generalized Pigeonhole Principle: If n pigeonholes are occupied by kn + 1 pigeons, then at least one pigeonhole is occupied by k + 1 or more pigeons.

Example 1: Birthmonth

- In a group of 13 people, we have 2 or more who are born in the same month.

# pigeons	# holes	At least # born in the same month
13	12	2 or more
20	12	2 or more
121	12	11 or more
65	12	6 or more
111	12	10 or more
$\geq kn+1$	n	$k+1$ or more

Example 2: Handshaking

Given a group of n people (n>1), each shakes hands with some (a nonzero number of) people in the group. We can find at least two who shake hands with the same number of people.

Proof:
Number of pigeons (number of people): n
Number of pigeonholes (range of number of handshakes): n-1

Example 3: Cast in theater

A theater performs 7 plays in one season. There are 15 women. Then some play has at least 3 women in its cast.

Number of pigeons: 15
Number of pigeonholes: 7
$k*n+1=2*7+1$
3 or more pigeons in the same pigeonhole

Example 4: Pairwise difference

Given 8 different natural numbers, none greater than 14. Show that at least three pairs of them have the same difference.

Try a set: 1, 2, 3, 7, 9, 11, 12, 14
Difference of 12 and 14 = 2.
Same for 9 and 11; 7 and 9; 1 and 3.
In this set, there are four pairs that all have the same difference.

Example 4: Pairwise difference

Given 8 different natural numbers, none greater than 14.
Show that at least three pairs of them have the same difference.

Try a set: 1, 2, 3, 7, 9, 11, 12, 14
Difference of 12 and 14 = 2.
Same for 9 and 11; 7 and 9; 1 and 3.
In this set, there are four pairs that all have the same difference.

Proof:
pigeons (different pairs: C(8,2) = 8*7/2): 28
pigeonholes (14-1): 13
Since $28 \geq k*n+1 = 2*13+1$, we have
3 or more pigeons in the same pigeonhole.

Remark

- Pigeonhole principle has applications to assignment and counting.
- The usage of the principle relies on the identification of the pigeons and the pigeonholes.

Outline

- The Pigeonhole Principle
- **Permutations and Combinations**

Permutations

Definition: A **permutation** of a set S is a sequence that contains every element of S exactly once.

For example, here are all six permutations of the set {a, b, c}:

(a, b, c) (a, c, b) (b, a, c)

(b, c, a) (c, a, b) (c, b, a)

Ordering is important here.

How many permutations of an n-element set are there?

You can think of a permutation as a ranking of the elements.

So the above question is asking how many rankings of an n-element set.

Permutations

How many permutations of an n-element set are there?

- There are n choices for the first element.
- For each of these, there are $n - 1$ remaining choices for the second element.
- For every combination of the first two elements, there are $n - 2$ ways to choose the third element, and so forth.
- Thus, there are a total of

$$n \cdot (n-1) \cdot (n-2) \cdot \cdot \cdot 3 \cdot 2 \cdot 1 = n!$$

This is called n **factorial**.

permutations of an n-element set.

Stirling's formula (optional):

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Example

Suppose each digit is an element in {1,2,3,4,5,6,7,8,9}.
How many 9-digit numbers are there where each nonzero digit appears once?

Each such number corresponds to a permutation of 123456789,

and each permutation corresponds to such a number.

So the numbers of such numbers is equal to

the number of permutations of {1,2,3,4,5,6,7,8,9}.

Hence there are exactly 9! such numbers.

Alternatively, one can use the generalized product rule directly to obtain the same result.

Combinations

How many subsets of size k of an n-element set?

Consider the set {1,2,3,4,5} where n=5.

If k=2, then there are 10 possible subsets of size 2,

i.e. {1,2}, {1,3}, {1,4}, {1,5}, {2,3}, {2,4}, {2,5}, {3,4}, {3,5}, {4,5}.

If k=3, then there are also 10 possible subsets of size 3,

i.e. {1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5}

{1,4,5}, {2,3,4}, {2,3,5}, {2,4,5}, {3,4,5}

Ordering is NOT important here.

Combinations

How many subsets of size k of an n-element set?

- There are n choices for the first element.
- For each of these, there are $n - 1$ remaining choices for the second element.
- There are $n - k + 1$ remaining choices for the last element.
- Thus, there are a total of
  $n \cdot (n - 1) \cdot (n - 2) \cdots (n - k + 1)$ to choose k elements.

So far we counted the number of ways to choose k elements, when the ordering is important.

e.g. {1,2,3}, {1,3,2}, {2,1,3}, {2,3,1}, {3,1,2}, {3,2,1} will be counted as 6 different ways.

Combinations

How many subsets of size k of an n-element set?

We form the subsets by picking one element at a time.

- There are n choices for the first element.
- For each of these, there are $n - 1$ remaining choices for the second element.
- There are $n - k + 1$ remaining choices for the last element.
- Thus, there are a total of

  $n \cdot (n - 1) \cdot (n - 2) \cdots (n - k + 1)$ to choose k elements.

So far we counted the number of ways to choose k elements, when the ordering is important.

Combinations

How many subsets of size k of an n-element set?

- Thus, there are a total of

  $n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$ ways to choose k elements,

  when the ordering is important.

How many different ordering of k elements are (over)-counted?

e.g. If we are forming subsets of size 3, then

(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)

are counted as 6 different ways if the ordering is important.

In general, each subset of size k has k! different orderings,
and so each subset is counted k! times in the above way of choosing k elements.

Combinations

How many subsets of size k of an n-element set?

- Thus, there are a total of

  $n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$ ways to choose k elements,

  when the ordering is important.

- Each subset is counted, but is counted k! times, because

  each subset contributes k! different orderings to the above.

- So, when the ordering is not important, the answer is:

This is the shorthand for "n choose k"

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

Example: Team Formation

There are m boys and n girls.

How many ways are there to form a team with 3 boys and 3 girls?

There are $\binom{m}{3}$ choices of 3 boys and $\binom{n}{3}$ choices for 3 girls.

So by the product rule there are $\binom{m}{3}\binom{n}{3}$ choices of such a team.

If $m < 3$ or $n < 3$, then the answer should be zero.

Don't worry. We don't like to trick you this way.

Example: Bit Strings with k Zeros

How many n-bit sequences contain k zeros and (n − k) ones?

We can think of this problem as choosing k positions (out of the n possible positions) and set them to zeroes and set the remaining positions to ones.

So the above question is asking the number of possible positions of the k zeros, and the answer is:

$$\binom{n}{k}$$

Example: Unbalanced Bit Strings

We say a bit string is unbalanced if there are more ones than zeroes or more zeros than ones.

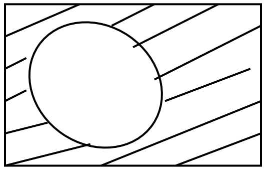
How many n-bit strings are unbalanced?

If n is odd, then every n-bit string is unbalanced, and the answer is 2^n.

If n is even, then the number of balanced strings is

$\binom{n}{n/2}$ by choosing n/2 positions to zeroes.

So the number of unbalanced n-bit strings is equal to the number of all n-bit strings minus the number of balanced strings, and so the answer is $2^n - \binom{n}{n/2}$ (counting the complement)



Poker Hands

There are 52 cards in a deck.

Each card has a suit and a value.

4 suits (♠ ♥ ♦ ♣)

13 values (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)

Five-Card Draw is a card game in which each player is initially dealt a hand, a subset of 5 cards.

How many different hands?

$$\binom{52}{5} = 2598960$$

Example 1: Four of a Kind

A Four-of-a-Kind is a set of four cards with the same value.

$$\{ \quad 8\spadesuit, \quad 8\diamondsuit, \quad Q\heartsuit, \quad 8\heartsuit, \quad 8\clubsuit \quad \}$$
$$\{ \quad A\clubsuit, \quad 2\clubsuit, \quad 2\heartsuit, \quad 2\diamondsuit, \quad 2\spadesuit \quad \}$$

How many different hands contain a Four-of-a-Kind?

One way to do this is to first map the problem into a problem of counting sequences.

Example 1: Four of a Kind

A hand with a Four-of-a-Kind is completely described by a sequence specifying:

1. The value of the four cards.
2. The value of the extra card.
3. The suit of the extra card.

$$(8, Q, \heartsuit) \quad \leftrightarrow \quad \{ \quad 8\spadesuit, \quad 8\diamondsuit, \quad 8\heartsuit, \quad 8\clubsuit, \quad Q\heartsuit \quad \}$$
$$(2, A, \clubsuit) \quad \leftrightarrow \quad \{ \quad 2\clubsuit, \quad 2\heartsuit, \quad 2\diamondsuit, \quad 2\spadesuit, \quad A\clubsuit \quad \}$$

There are 13 choices for (1), 12 choices for (2), and 4 choices for (3).
By generalized product rule, there are 13x12x4 = 624 hands.

Only 1 hand in about 4165 has a Four-of-a-Kind!

Example 2: Full House

A Full House is a hand with three cards of one value and two cards of another value.

$$\{\quad 2\spadesuit,\quad 2\clubsuit,\quad 2\diamondsuit,\quad J\clubsuit,\quad J\diamondsuit\quad\}$$
$$\{\quad 5\diamondsuit,\quad 5\clubsuit,\quad 5\heartsuit,\quad 7\heartsuit,\quad 7\clubsuit\quad\}$$

How many different hands contain a Full House?

Example 2: Full House

There is a bijection between Full Houses and sequences specifying:

1. The value of the triple, which can be chosen in 13 ways.
2. The suits of the triple, which can be selected in (4 3) ways.
3. The value of the pair, which can be chosen in 12 ways.
4. The suits of the pair, which can be selected in (4 2) ways.

$$(2, \{\spadesuit, \clubsuit, \diamondsuit\}, J, \{\clubsuit, \diamondsuit\}) \leftrightarrow \{\ 2\spadesuit,\ 2\clubsuit,\ 2\diamondsuit,\ J\clubsuit,\ J\diamondsuit\ \}$$
$$(5, \{\diamondsuit, \clubsuit, \heartsuit\}, 7, \{\heartsuit, \clubsuit\}) \leftrightarrow \{\ 5\diamondsuit,\ 5\clubsuit,\ 5\heartsuit,\ 7\heartsuit,\ 7\clubsuit\ \}$$

By generalized product rule, there are

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3744$$

Only 1 hand in about 634 has a Full House!

Example 3: Two Pairs

How many hands have Two Pairs; that is,
two cards of one value, two cards of another value,
and one card of a third value?

$$\{\quad 3\diamondsuit,\quad 3\spadesuit,\quad Q\diamondsuit,\quad Q\heartsuit,\quad A\clubsuit \quad\}$$
$$\{\quad 9\heartsuit,\quad 9\diamondsuit,\quad 5\heartsuit,\quad 5\clubsuit,\quad K\spadesuit \quad\}$$

Example 3: Two Pairs

1. The value of the first pair, which can be chosen in 13 ways.
2. The suits of the first pair, which can be selected (4 2) ways.
3. The value of the second pair, which can be chosen in 12 ways.
4. The suits of the second pair, which can be selected in (4 2) ways
5. The value of the extra card, which can be chosen in 11 ways.
6. The suit of the extra card, which can be selected in 4 ways.

Number of Two pairs = $13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4$

Double Count!

$$(3, \{\diamondsuit, \spadesuit\}, Q, \{\diamondsuit, \heartsuit\}, A, \clubsuit) \searrow$$

$$\{\ 3\diamondsuit,\ 3\spadesuit,\ Q\diamondsuit,\ Q\heartsuit,\ A\clubsuit\ \}$$

$$(Q, \{\diamondsuit, \heartsuit\}, 3, \{\diamondsuit, \spadesuit\}, A, \clubsuit) \nearrow$$

So the answer is $\frac{1}{2} \cdot 13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4 = 123552$

Example 4: Every Suit

How many hands contain at least one card from every suit?

$$\{ \; 7\diamondsuit, \; K\clubsuit, \; 3\diamondsuit, \; A\heartsuit, \; 2\spadesuit \; \}$$

1. The value of each suit, which can be selected in 13x13x13x13 ways.
2. The suit of the extra card, which can be selected in 4 ways.
3. The value of the extra card, which can be selected in 12 ways.

$$(7, K, A, 2, \diamondsuit, 3) \;\leftrightarrow\; \{ \; 7\diamondsuit, \; K\clubsuit, \; A\heartsuit, \; 2\spadesuit, \; 3\diamondsuit \; \}$$

$$(7, K, A, 2, \diamondsuit, 3) \searrow$$

$$\{ \; 7\diamondsuit, \; K\clubsuit, \; A\heartsuit, \; 2\spadesuit, \; 3\diamondsuit \; \}$$

$$(3, K, A, 2, \diamondsuit, 7) \nearrow$$

Double count!

So the answer is 13^4x4x12/2 = 685464

Next class

- Topic: Binomial Coefficients and Identities
- Pre-class reading: Chap 6.4