Lecture 27: Binomial Coefficients and Identities

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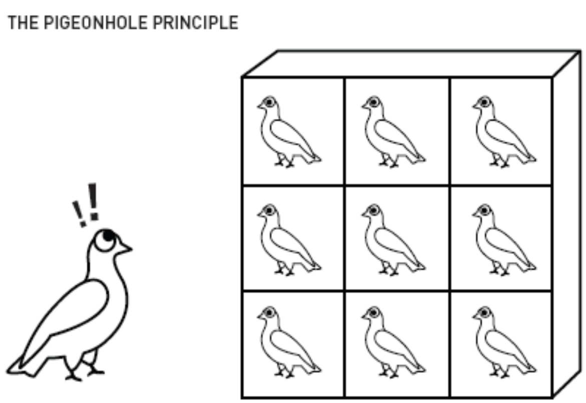
Recap Previous Lecture

- Pigeonhole Principle
- Permutation and combination



$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

Outline

- r-Permutations and r-Combinations
- Permutations and Combinations with Repetition
- Binomial coefficients, combinatorial proof

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- **r-Permutations and r-Combinations**
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r-permutations

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of a set with n distinct elements.

- How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution: $P(100, 3) = 100 \cdot 99 \cdot 98 = 970{,}200$.

- How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: $P(6,6) = 6! = 720$
ABC, D, E, F, G, H

Example

- Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

$$P(8, 3) = 8 \cdot 7 \cdot 6 = 336$$

- Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

r-combination

- An ***r*-combination** of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

$$P(n, r) = C(n, r) \cdot P(r, r).$$

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!\,(n-r)!}.$$

$$C(n, r) = \frac{n!}{r!\,(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

Example

- How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

$$C(52, 5) = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2{,}598{,}960.$$

$$C(52, 47) = \frac{52!}{47!5!}$$

Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Example

- A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

$$C(30, 6) = \frac{30!}{6!\,24!}$$

$$= \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 593{,}775.$$

Example

- Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

$$C(9,3) \cdot C(11,4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27{,}720.$$

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- r-Permutations and r-Combinations
- **Permutations and Combinations with Repetition**
- Binomial coefficients, combinatorial proof

Permutations and Combination with Repetition

THEOREM 1

The number of r-permutations of a set of n objects with repetition allowed is n^r.

THEOREM 2

There are $C(n + r - 1, r) = C(n + r - 1, n - 1)$ r-combinations from a set with n elements when repetition of elements is allowed.

A 6-combination of a set with four elements

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containing exactly two of the first element, one of the second element, none of the third element, and three of the fourth element of the set.

n – 1 bars and r stars corresponds to an r-combination of the set with n elements

Example

- Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

$$C(4+6-1, 6) = C(9, 6)$$

$$C(9, 6) = C(9, 3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$$

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- Permutations and Combinations with Repetition
- **Binomial coefficients, combinatorial proof**

Binomial Theorem

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

We can compute the coefficients by simple counting arguments.

$$(1+x)^n = \underbrace{(1+x)(1+x)(1+x)\cdots\cdots(1+x)}_{\text{n times}}$$

Each term corresponds to selecting 1 or x from each of the n factors.

c_k is number of terms with exactly k x's are selected from n factors.

$$c_k = \binom{n}{k}$$

Binomial Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

$(1+X)^0 =$ 1

$(1+X)^1 =$ 1 + 1X

$(1+X)^2 =$ 1 + 2X + 1X^2

$(1+X)^3 =$ 1 + 3X + 3X^2 + 1X^3

$(1+X)^4 =$ 1 + 4X + 6X^2 + 4X^3 + 1X^4

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Binomial Coefficients

In general we have the following identity:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

When x=1, y=1, it says that $2^n = \sum_{i=0}^{n} \binom{n}{i}$

When x=1, y=-1, it says that

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} + \ldots$$

$$\Rightarrow \sum_{i \text{ odd}} \binom{n}{i} = \sum_{j \text{ even}} \binom{n}{j}$$

Proving Identities

$$\binom{n}{k} = \binom{n}{n-k}$$

Direct proof: $$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Combinatorial proof: Number of ways to choose k items from n items

= number of ways to choose n-k items from n items

Finding a Combinatorial Proof

A **combinatorial proof** is an argument that establishes an algebraic fact by relying on counting principles.

Many such proofs follow the same basic outline:

1. Define a set S.
2. Show that |S| = n by counting one way.
3. Show that |S| = m by counting another way.
4. Conclude that n = m.

Double counting

Proving Identities

Pascal's Formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Direct proof:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!}$$

$$= \frac{n!(n+1)}{k!(n-k+1)!}$$

$$= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$$

Proving Identities

Pascal's Formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Combinatorial proof:

- The LHS is number of ways to choose k elements from n+1 elements.
- Let the first element be x.
- If we choose x, then we need to choose k-1 elements from the remaining n elements, and number of ways to do so is $\binom{n}{k-1}$
- If we don't choose x, then we need to choose k elements from the remaining n elements, and number of ways to do so is $\binom{n}{k}$
- This partitions the ways to choose k elements from n+1 elements, therefore $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$
$$\cdots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$
$$\cdots$$

(b)

Combinatorial Proof

$$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Consider we have 2n balls, n of them are red, and n of them are blue.

The RHS is number of ways to choose n balls from the 2n balls.
To choose n balls, we can

- choose 0 red ball and n blue balls, number of ways = $\binom{n}{0}\binom{n}{n}$
- choose 1 red ball and n-1 blue balls, number of ways = $\binom{n}{1}\binom{n}{n-1}$
- …
- choose i red balls and n-i blue balls, number of ways = $\binom{n}{i}\binom{n}{n-i}$
- …
- choose n red balls and 0 blue ball, number of ways = $\binom{n}{n}\binom{n}{0}$

Hence number of ways to choose n balls is also equal to $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$

Another Way to Combinatorial Proof (Optional)

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$$

We can also prove the identity by comparing a coefficient of two polynomials.

Consider the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$

Consider the coefficient of x^n in these two polynomials.

Clearly the coefficient of x^n in $(1+x)^{2n}$ is equal to the RHS.

$$(1+x)^n(1+x)^n = (\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{n}x^n)(\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{n}x^n)$$

So the coefficient of x^n in $(1+x)^n(1+x)^n$ is equal to the LHS.

More Combinatorial Proof

$$\sum_{r=0}^{n} \binom{n}{r}\binom{2n}{n-r} = \binom{3n}{n}$$

Let S be all n-card hands that can be dealt from a deck containing
n red cards (numbered 1, . . . , n) and 2n black cards (numbered 1, . . . , 2n).

The right hand side = # of ways to choose n cards from these 3n cards.

The left hand side = # of ways to choose r cards from red cards x
# of ways to choose n-r cards from black cards
= # of ways to choose n cards from these 3n cards
= the right hand side.

Exercises

Prove that

$$3^n = 1+2n+4\binom{n}{2}+8\binom{n}{3}+\ldots+2^k\binom{n}{k}+\ldots+2^n\binom{n}{n}$$

Give a combinatorial proof of the following identify.

$$\binom{n}{0}\binom{2n}{n}+\binom{n}{1}\binom{2n}{n-1}+\ldots+\binom{n}{k}\binom{2n}{n-k}+\ldots+\binom{n}{n}\binom{2n}{0}=\binom{3n}{n}$$

Can you give a direct proof of it?

Quick Summary

We have studied how to determine the size of a set directly.

The basic rules are the sum rule, product rule, and the generalized product rule.

We apply these rules in counting permutations and combinations,

which are then used to count other objects like poker hands.

Then we prove the binomial theorem and study combinatorial proofs of identities.

Finally we learn the inclusion-exclusion principle and see some applications.

Later we will learn how to count “indirectly” by “mapping”.

Next class

- Topic: Inclusion-exclusion Principle
- Pre-class reading: Chap 8.5-8.6