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Lecture 28: Inclusion-exclusion Principle

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Recap Previous Lecture

- r-permutation and r-combination
- Permutation and combination with Repetition
- Binomial coefficients, combinatorial proof

$$P(n, r) = C(n, r) \cdot P(r, r).$$

$$C(n + r - 1, r) = C(n + r - 1, n - 1)$$

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Outline

- Combinatorial Proof
- Inclusion-exclusion Principle

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- Combinatorial Proof
- **Inclusion-exclusion Principle**

Finding a Combinatorial Proof

A **combinatorial proof** is an argument that establishes an algebraic fact by relying on counting principles.

Many such proofs follow the same basic outline:

1. Define a set S.
2. Show that |S| = n by counting one way.
3. Show that |S| = m by counting another way.
4. Conclude that n = m.

Double counting

Proving Identities

Pascal's Formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Direct proof:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!}$$

$$= \frac{n!(n+1)}{k!(n-k+1)!}$$

$$= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$$

Combinatorial Proof

$$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Consider we have 2n balls, n of them are red, and n of them are blue.

The RHS is number of ways to choose n balls from the 2n balls.
To choose n balls, we can

- choose 0 red ball and n blue balls, number of ways = $\binom{n}{0}\binom{n}{n}$
- choose 1 red ball and n-1 blue balls, number of ways = $\binom{n}{1}\binom{n}{n-1}$
- …
- choose i red balls and n-i blue balls, number of ways = $\binom{n}{i}\binom{n}{n-i}$
- …
- choose n red balls and 0 blue ball, number of ways = $\binom{n}{n}\binom{n}{0}$

Hence number of ways to choose n balls is also equal to $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$

Another Way to Combinatorial Proof (Optional)

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$$

We can also prove the identity by comparing a coefficient of two polynomials.

Consider the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$

Consider the coefficient of x^n in these two polynomials.

Clearly the coefficient of x^n in $(1+x)^{2n}$ is equal to the RHS.

$$(1+x)^n(1+x)^n = (\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{n}x^n)(\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{n}x^n)$$

So the coefficient of x^n in $(1+x)^n(1+x)^n$ is equal to the LHS.

More Combinatorial Proof

$$\sum_{r=0}^{n} \binom{n}{r}\binom{2n}{n-r} = \binom{3n}{n}$$

Let S be all n-card hands that can be dealt from a deck containing
n red cards (numbered 1, . . . , n) and 2n black cards (numbered 1, . . . , 2n).

The right hand side = # of ways to choose n cards from these 3n cards.

The left hand side = # of ways to choose r cards from red cards x
# of ways to choose n-r cards from black cards
= # of ways to choose n cards from these 3n cards
= the right hand side.

Exercises

Prove that

$$3^n = 1 + 2n + 4\binom{n}{2} + 8\binom{n}{3} + \ldots + 2^k\binom{n}{k} + \ldots + 2^n\binom{n}{n}$$

Give a combinatorial proof of the following identify.

$$\binom{n}{0}\binom{2n}{n} + \binom{n}{1}\binom{2n}{n-1} + \ldots + \binom{n}{k}\binom{2n}{n-k} + \ldots + \binom{n}{n}\binom{2n}{0} = \binom{3n}{n}$$

Can you give a direct proof of it?

Quick Summary

We have studied how to determine the size of a set directly.

The basic rules are the sum rule, product rule, and the generalized product rule.

We apply these rules in counting permutations and combinations,

which are then used to count other objects like poker hands.

Then we prove the binomial theorem and study combinatorial proofs of identities.

Finally we learn the inclusion-exclusion principle and see some applications.

Later we will learn how to count "indirectly" by "mapping".

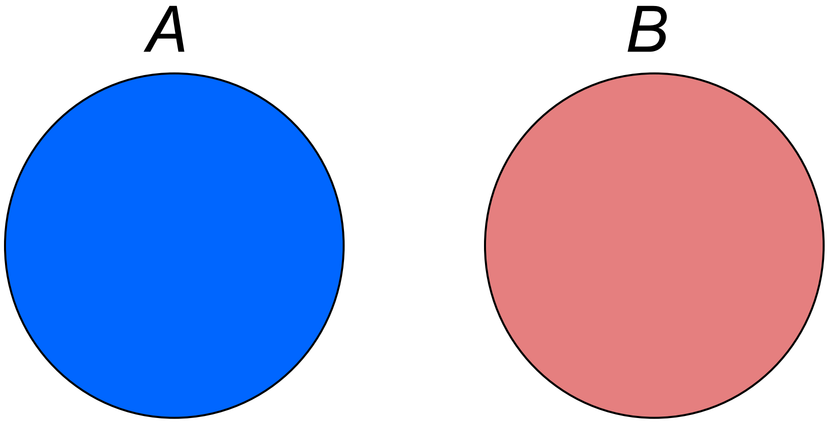
Outline

- Combinatorial Proof
- **Inclusion-exclusion Principle**

Sum Rule

If sets A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

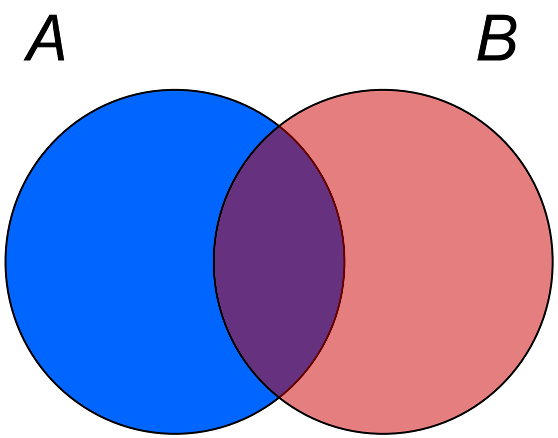


What if A and B are not disjoint?

Inclusion-Exclusion (2 Sets)

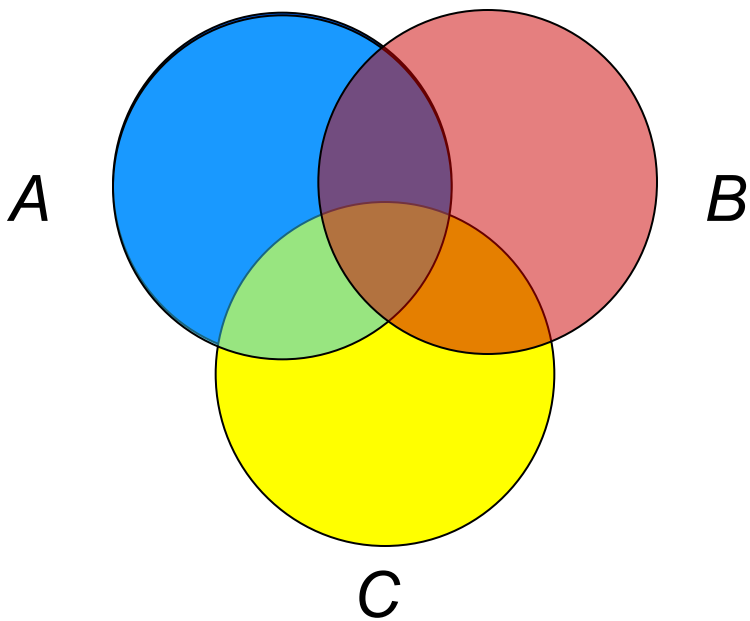
For two arbitrary sets *A* and *B*

$$|A \cup B| = |A| + |B| - |A \cap B|$$



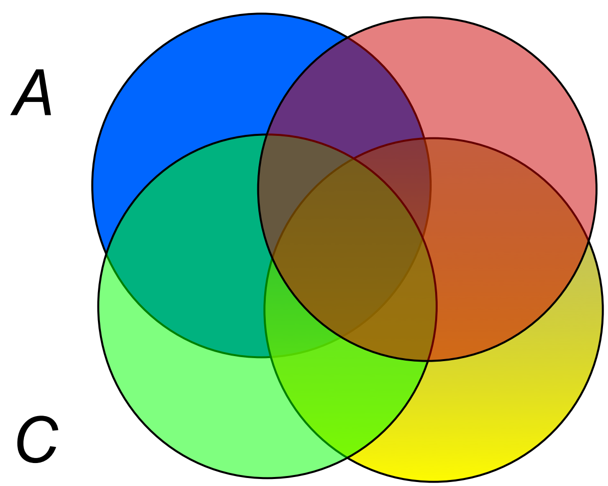
Inclusion-Exclusion (3 Sets)

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$



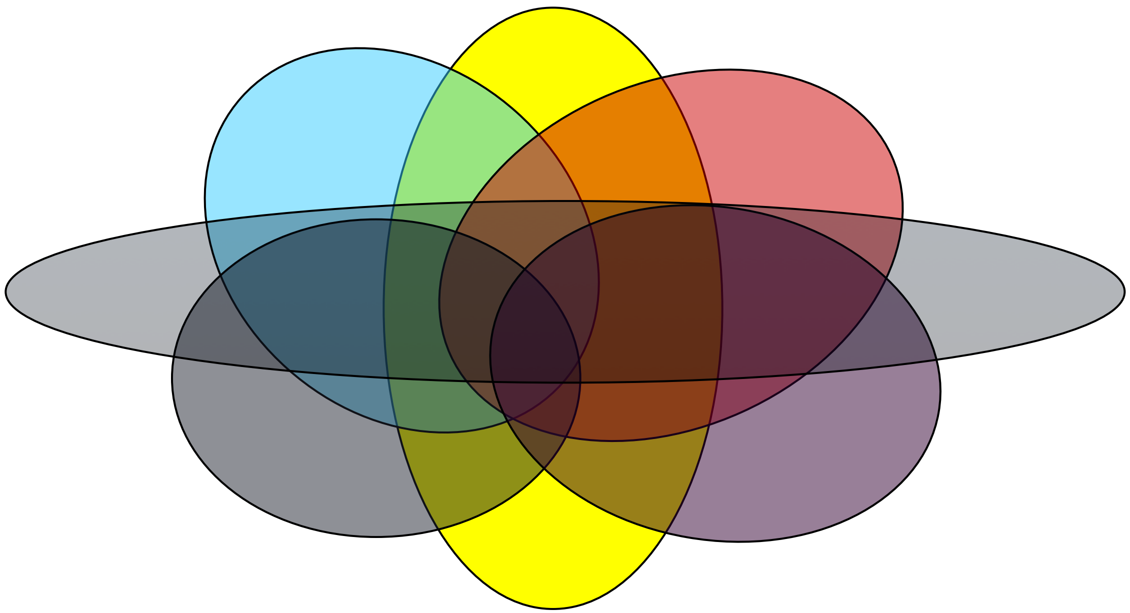
Inclusion-Exclusion (4 Sets)

$$
\begin{aligned}
|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\
& - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
& + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\
& - |A \cap B \cap C \cap D|
\end{aligned}
$$



Inclusion-Exclusion (n Sets)

What is the inclusion-exclusion formula for the union of n sets?



Inclusion-Exclusion (n Sets)

$$\left|A_1 \cup A_2 \cup \cdots \cup A_n\right| =$$

sum of sizes of all single sets
– sum of sizes of all 2-set intersections
+ sum of sizes of all 3-set intersections
– sum of sizes of all 4-set intersections
…
+ $(-1)^{n+1} \times$ sum of sizes of intersections of all n sets

$$= \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ |S|=k}} \left| \bigcap_{i \in S} A_i \right|$$

Inclusion-Exclusion (n Sets)

$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n|$

sum of sizes of all single sets

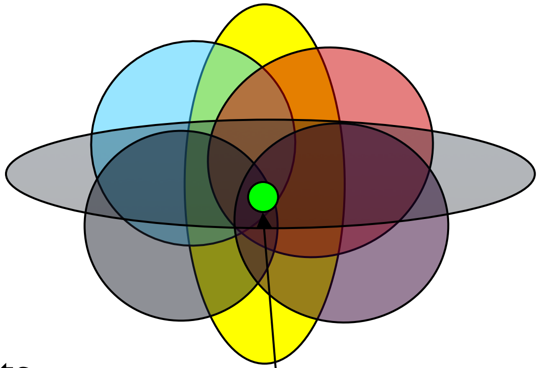
– sum of sizes of all 2-set intersections

\+ sum of sizes of all 3-set intersections

– sum of sizes of all 4-set intersections

…

\+ $(-1)^{n+1} \times$ sum of sizes of intersections of all *n* sets



We want to show that every element is counted exactly once.

Consider an element which belongs to exactly k sets, say $A_1, A_2, A_3, \ldots, A_k$.

In the formula, such an element is counted the following number of times:

$$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \binom{k}{4} + \ldots + (-1)^{k+1}\binom{k}{k} = 1$$

Therefore each element is counted exactly once, and thus the formula is correct

Inclusion-Exclusion (n Sets)

$$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \binom{k}{4} + \ldots + (-1)^{k+1}\binom{k}{k} = 1$$

$$(x+y)^k = \sum_{k=0}^{n} \binom{k}{i} x^i y^{k-i}$$

Plug in x=1 and y=-1 in the above binomial theorem, we have

$$0 = \binom{k}{0} - \binom{k}{1} + \binom{k}{2} + \ldots + (-1)^i \binom{k}{i} + \ldots + (-1)^k \binom{k}{k}$$

$$\Rightarrow \binom{k}{1} - \binom{k}{2} + \ldots + (-1)^{i+1}\binom{k}{i} + \ldots + (-1)^{k+1}\binom{k}{k} = \binom{k}{0}$$

$$= 1$$

Christmas Party

In a Christmas party, everyone brings his/her present.

There are n people and so there are totally n presents.

Suppose the host collects and shuffles all the presents.

Now everyone picks a random present.

What is the probability that no one picks his/her own present?

Let the n presents be {1, 2, 3, …, n}, where the present i is owned by person i.

Now a random ordering of the presents means a permutation of {1, 2, 3, …, n}.

e.g. (3,2,1) means the person 1 picks present 3, person 2 picks present 2, etc.

And the question whether someone picks his/her own present becomes

whether there is a number i which is in position i of the permutation.

Fixed Points in a Permutation

Given a random permutation of {1, 2, 3, …, n},
what is the probability that a permutation has no fixed point
(i.e number i is not in position i for all i)?

e.g. {2, 3, 1, 5, 6, 4} has no fixed point,
{3, 4, 7, 5, 2, **6**, 1} has a fixed point,
{5, 4, **3**, 2, 1} has a fixed point.

You may wonder why we are suddenly asking a probability question.
Actually, this is equivalent to the following counting question:

What is the number of permutations of {1,2,3,…,n} with no fixed point?

Fixed Points in a Permutation

What is the number of permutations of {1,2,3,…,n} with no fixed point?

For this question, it is more convenient to count the complement.

Let S be the set of permutations of {1,2,3…,n} with some fixed point(s).

Let A_1 be the set of permutations in which the number 1 is in position 1.
…
Let A_j be the set of permutations in which the number j is in position j.
…
Let A_n be the set of permutations in which the number n is in position n.

$S = A_1 \cup A_2 \cup \ldots \cup A_n$

Note that A_i and A_j are not disjoint, and so we need inclusion-exclusion.

Fixed Points in a Permutation

Let S be the set of permutations of {1,2,3…,n} with some fixed point(s).

Let A_j be the set of permutations in which the number j is in position j.

$$S = A_1 \cup A_2 \cup \ldots \cup A_n$$

How large is $|A_j|$?

Once we fixed j, we can have any permutation on the remaining n-1 elements.

Therefore, $|A_j| = (n-1)!$

How large is $|A_i \cap A_j|$?

Once we fixed i and j, we can have any permutation on the remaining n-2 elements.

Therefore, $|A_i \cap A_j| = (n-2)!$

Fixed Points in a Permutation

Let S be the set of permutations of {1,2,3…,n} with some fixed point(s).

Let A_j be the set of permutations in which the number j is in position j.

$$S = A_1 \cup A_2 \cup \ldots \cup A_n$$

How large is the intersection of k sets?

In the intersection of k sets, there are k positions being fixed.

Then we can have any permutation on the remaining n-k elements.

Therefore, |the intersection of k sets| = (n-k)!

Fixed Points in a Permutation

Let S be the set of permutations of {1,2,3…,n} with some fixed point(s).

Let A_j be the set of permutations in which the number j is in position j.

$S = A_1 \cup A_2 \cup \ldots \cup A_n$

|the intersection of k sets| = (n-k)!

$$|S| = |A_1 \cup A_2 \cup \ldots \cup A_n|$$
$$= \binom{n}{1}(n-1)!$$
$$-\binom{n}{2}(n-2)!$$
$$+\binom{n}{3}(n-3)!$$
$$\cdots$$
$$+(-1)^{n+1}\binom{n}{n}(n-n)!$$

$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n|$

sum of sizes of all single sets
– sum of sizes of all 2-set intersections
\+ sum of sizes of all 3-set intersections
– sum of sizes of all 4-set intersections
…
\+ $(-1)^{n+1} \times$ sum of sizes of intersections of n sets

Fixed Points in a Permutation

Let S be the set of permutations of {1,2,3…,n} with some fixed point(s).

Let A_j be the set of permutations in which the number j is in position j.

$$S = A_1 \cup A_2 \cup \ldots \cup A_n$$

|the intersection of k sets| = (n-k)!

$$|S| = |A_1 \cup A_2 \cup \ldots \cup A_n|$$

$$= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! \ldots + (-1)^{n+1}\binom{n}{n}(n-n)!$$

$$|S| = |A_1 \cup A_2 \cup \ldots \cup A_n|$$

$$= n! - n!/2! + n!/3! + \ldots (-1)^{i+1} n!/i! + \ldots + (-1)^{n+1}$$

Fixed Points in a Permutation

Let S be the set of permutations of {1,2,3…,n} with some fixed point(s).

Let A_j be the set of permutations in which the number j is in position j.

$$S = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$|S| = n! - n!/2! + n!/3! + \ldots (-1)^{i+1} n!/i! + \ldots + (-1)^{n+1}$$

The number of permutations with no fixed points

$= n! - |S|$

$= n! - n! + n!/2! - n!/3! + \ldots (-1)^i n!/i! + \ldots + (-1)^n$

$= n! (1 - 1/1! + 1/2! - 1/3! + \ldots + (-1)^i 1/i! \ldots + (-1)^n 1/n!)$

-> n!/e (where e is the constant 2.71828…)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum^{\infty} \frac{x^n}{n!}$$

Next class

- Topic: Probability
- Pre-class reading: Chap 7.1