Lecture 3: Predicate Logic and Quantifiers

Dr. Chengjiang Long
Computer Vision Researcher at Kitware Inc.
Adjunct Professor at SUNY at Albany.
Email: **clong2@albany.edu**

Recap Previous Lecture

1. Definition of proposition. Opinion, interrogative, and imperative statements are not propositions.
2. Logical connects:
   - Negation, logical conjunction (and), logical disjunction (or), exclusive or (xor), implication, and biconditional.
3. Inverse, converse, and contrapositive
4. True table.
5. Logical equivalence

p	$\neg p$
0	1
1	0

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$P \rightarrow q$	$P <-> q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Recap Previous Lecture

- Use true table to prove logical equivalence.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \vee (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$
0	0	0	1	1	1	0	1
0	0	1	1	1	1	0	1
0	1	0	1	0	1	0	1
0	1	1	1	1	1	0	1
1	0	0	0	1	1	0	1
1	0	1	1	1	1	0	1
1	1	0	0	0	0	1	0
1	1	1	1	1	1	1	1

Outline

- Logical Equivalences
- Introduction to Predicate Logic
- Quantifiers

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- **Logical Equivalences**
- Introduction to Predicate Logic
- Quantifiers

Logical Equivalences: Cheat Sheet (1)

Identities (Equivalences)	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

Logical Equivalences: Cheat Sheet (2)

Logical Equivalences Involving Conditional Statements.
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Using Logical Equivalences: Example 1

- Logical equivalences can be used to construct additional logical equivalences
- Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

0. $(p \wedge q) \rightarrow q$
1. $\equiv \neg(p \wedge q) \vee q$ — Implication Law on 0
2. $\equiv (\neg p \vee \neg q) \vee q$ — De Morgan's Law (1^{st}) on 1
3. $\equiv \neg p \vee (\neg q \vee q)$ — Associative Law on 2
4. $\equiv \neg p \vee 1$ — Negation Law on 3
5. $\equiv 1$ — Domination Law on 4

Using Logical Equivalences: Example 3

- Example (Exercise 17)*: Show that $\neg(p \leftrightarrow q) \equiv (p \leftrightarrow \neg q)$
- Sometimes it helps to start with the second proposition $(p \leftrightarrow \neg q)$

0.	$(p \leftrightarrow \neg q)$	
1.	$\equiv (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$	Equivalence Law on 0
2.	$\equiv (\neg p \vee \neg q) \wedge (q \vee p)$	Implication Law on 1
3.	$\equiv \neg(\neg((\neg p \vee \neg q) \wedge (q \vee p)))$	Double negation on 2
4.	$\equiv \neg(\neg(\neg p \vee \neg q) \vee \neg(q \vee p))$	De Morgan's Law…
5.	$\equiv \neg((p \wedge q) \vee (\neg q \wedge \neg p))$	De Morgan's Law
6.	$\equiv \neg((p \vee \neg q) \wedge (p \vee \neg p) \wedge (q \vee \neg q) \wedge (q \vee \neg p))$	Distribution Law
7.	$\equiv \neg((p \vee \neg q) \wedge (q \vee \neg p))$	Identity Law
8.	$\equiv \neg((q \rightarrow p) \wedge (p \rightarrow q))$	Implication Law
9.	$\equiv \neg(p \leftrightarrow q)$	Equivalence Law

*See Table 8 (p 25) but you are not allowed to use the table for the proof

Using Logical Equivalences: Example 3

- Show that $\neg(q \to p) \vee (p \wedge q) \equiv$ q

0. $\neg(q \to p) \vee (p \wedge q)$
1. $\equiv \neg(\neg q \vee p) \vee (p \wedge q)$ — Implication Law
2. $\equiv (q \wedge \neg p) \vee (p \wedge q)$ — De Morgan's & Double negation
3. $\equiv (q \wedge \neg p) \vee (q \wedge p)$ — Commutative Law
4. $\equiv q \wedge (\neg p \vee p)$ — Distributive Law
5. $\equiv q \wedge 1$ — Identity Law

   $\equiv q$ — Identity Law

Logic in Programming: Example 2 (revisited)

- Recall the loop

```
While
    ((i<size AND A[i]>10) OR
     (i<size AND A[i]<0) OR
     (i<size AND (NOT (A[i]!=0 AND NOT (A[i]>=10)))))
```

- Now, using logical equivalences, simplify it!
- Using De Morgan's Law and Distributivity

```
While ((i<size) AND
      ((A[i]>10 OR A[i]<0) OR
        (A[i]==0 OR A[i]>=10)))
```

- Noticing the ranges of the 4 conditions of A[i]

```
While ((i<size) AND (A[i]>=10 OR A[i]<=0))
```

Programming Pitfall Note

- In C, C++ and Java, applying the commutative law is not such a good idea.
- For example, consider accessing an integer array A of size n:

```
if (i<n && A[i]==0) i++;
```

is not equivalent to

```
if (A[i]==0 && i<n) i++;
```

Outline

- Logical Equivalences
- **Introduction to Predicate Logic**
- Quantifiers

Introduction

- Consider the statements:

  $x>3$, $x=y+3$, $x+y=z$

- The symbols >, +, = denote relations between *x* and 3, *x*, *y*, and 4, and *x*,*y*, and *z*, respectively
- These relations may hold or not hold depending on the values that *x*, *y*, and *z* may take.
- A **<u>predicate</u>** is a property that is affirmed or denied about the subject (in logic, we say '**variable**' or '**argument**') of a statement
- Consider the statement : '*x* is greater than 3'
  - '*x*' is the subject
  - 'is greater than 3' is the predicate

Propositional Functions (1)

- To write in Predicate Logic '*x* is greater than 3'
  - We introduce a functional symbol for the **predicate** and
  - Put the subject as an **argument** (to the functional symbol): $P(x)$
- Terminology
  - $P(x)$ is a statement
  - P is a predicate or propositional function
  - x as an argument

Propositional Functions (2)

- Examples:
  - Father(x): unary predicate
  - Brother(x.y): binary predicate
  - Sum(x,y,z): ternary predicate
  - P(x,y,z,t): n-ary predicate

Propositional Functions (3)

- **Definition:** A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional symbol P.
- Here: $(x_1, x_2, \ldots, x_n)$ is an n-tuple and P is a predicate
- We can think of a propositional function as a function that
  - Evaluates to true or false
  - Takes one or more arguments
  - Expresses a predicate involving the argument(s)
  - Becomes a proposition when values are assigned to the arguments

Propositional Functions: Example

- Let $Q(x,y,z)$ denote the statement '$x^2+y^2=z^2$'
  - What is the truth value of $Q(3,4,5)$?

    $Q(3,4,5)$ is true

  - What is the truth value of $Q(2,2,3)$?

    $Q(2,3,3)$ is false

  - How many values of (x,y,z) make the predicate true?

    There are infinitely many values that make the proposition true, how many right triangles are there?

Universe of Discourse

- Consider the statement ‘*x*>3’, does it make sense to assign to *x* the value ‘blue’?
- Intuitively, the **universe of discourse** is the set of all things we wish to talk about; that is the set of all objects that we can sensibly assign to a variable in a propositional function.
- What would be the universe of discourse for the propositional function below be:

Enrolled ICSI210(x)=‘x is enrolled in ICSI 210’

Universe of Discourse: Multivariate functions

- Each variable in an *n*-tuple (i.e., each argument) may have a different universe of discourse
- Consider an *n*-ary predicate *P*:

  $P(r,g,b,c)$= 'The *rgb*-values of the color *c* is (r,g,b)'
- Example, what is the truth value of
  - $P(255,0,0,red)$
  - $P(0,0,255,green)$
- What are the universes of discourse of (r,g,b,c)?

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Quantifiers: Introduction

- The statement '*x>3*' is not a proposition
- It becomes a proposition
  - When we assign values to the argument: '4>3' is false, '2<3' is true, or
  - When we quantify the statement
- Two quantifiers
  - Universal quantifier ∀
    $\forall$
    the proposition is true for all possible values in the universe of discourse
  - Existential quantifier∃
    $\exists$
    the proposition is true for some value(s) in the universe of discourse

Universal Quantifier: Definition

- **Definition**: The universal quantification of a predicate $P(x)$ is the proposition '$P(x)$ is true for all values of x in the universe of discourse.' We use the notation: $\forall\, x\, P(x)$, which is read 'for all x'.
- If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the universal quantifier is simply the conjunction of the propositions over all the elements

$$\forall\, x\, P(x) \Leftrightarrow P(n_1) \wedge P(n_2) \wedge \ldots \wedge P(n_k)$$

Universal Quantifier: Example 1

- Let $P(x)$: 'x must take a discrete mathematics course' and $Q(x)$: 'x is a CS student.'
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statements:
  - "Every CS student must take a discrete mathematics course."

    $$\forall x\ Q(x) \rightarrow P(x)$$
  - "Everybody must take a discrete mathematics course or be a CS student." $\forall x\ (P(x) \vee Q(x))$
  - "Everybody must take a discrete mathematics course and be a CS student." $\forall x\ (P(x) \wedge Q(x))$

Are these statements true or false?

Universal Quantifier: Example 2

- Express the statement: ‘for every *x* and every *y*, $x+y>10$’
- Answer:
  - Let $P(x,y)$ be the statement $x+y>10$
  - Where the universe of discourse for *x*, *y* is the set of integers
  - The statement is: $\forall x\ \forall y\ P(x,y)$
- Shorthand: $\forall x,y\ P(x,y)$

Existential Quantifier: Definition

- **Definition**: The existential quantification of a predicate $P(x)$ is the proposition 'There exists a value x in the universe of discourse such that $P(x)$ is true.' We use the notation: $\exists x\, P(x)$, which is read 'there exists x'.
- If the universe of discourse is finite, say $\{n_1,n_2,\ldots,n_k\}$, then the existential quantifier is simply the disjunction of the propositions over all the elements

$$\exists x\, P(x) \Leftrightarrow P(n_1) \vee P(n_2) \vee \ldots \vee P(n_k)$$

Existential Quantifier: Example 1

- Let $P(x,y)$ denote the statement 'x+y=5'
- What does the expression $\exists x\ \exists y\ P(x,y)$ mean?
- Which universe(s) of discourse make it true?

Existential Quantifier: Example 2

- Express the statement: ‘there exists a real solution to $ax^2+bx-c=0$’
- Answer:
  - Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
  - Where the universe of discourse for x is the set of real numbers. Note here that a, b, c are fixed constants.
  - The statement can be expressed as $\exists x\, P(x)$
- What is the truth value of $\exists x\, P(x)$?
  - It is false. When $b^2<4ac$, there are no real number x that can satisfy the predicate
- What can we do so that $\exists x\, P(x)$is true?
  - Change the universe of discourse to the complex numbers, $\mathbb{C}$

Quantifiers: Truth values

- In general, when are quantified statements true or false?

Statement	True when...	False when...
$\forall x\, P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false
$\exists x\, P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x

Mixing quantifiers (1)

- Existential and universal quantifiers can be used together to quantify a propositional predicate. For example:

$$\forall x \, \exists y \, P(x,y)$$

  is perfectly valid
- Alert:
  - The quantifiers must be read from left to right
  - The order of the quantifiers is important
  - $\forall x \, \exists y \, P(x,y)$ is not equivalent to $\exists y \, \forall x \, P(x,y)$

Mixing quantifiers (2)

- Consider
  - $\forall x\, \exists y\, Loves(x,y)$: Everybody loves somebody
  - $\exists y\, \forall x\, Loves(x,y)$: There is someone loved by everyone
- The two expressions do not mean the same thing
- $(\exists y\, \forall x\, Loves(x,y)) \rightarrow (\forall x\, \exists y\, Loves(x,y))$ but the converse does not hold
- However, you can commute similar quantifiers
  - $\forall x\, \forall y\, P(x,y)$ is equivalent to $\forall y\, \forall x\, P(x,y)$ (thus, $\forall x,y\, P(x,y)$)
  - $\exists x\, \exists y\, P(x,y)$ is equivalent to $\exists y\, \exists x\, P(x,y)$ (thus $\exists x,y\, P(x,y)$)

Mixing Quantifiers: Truth values

Statement	True when...	False when...
$\forall x \forall y\, P(x,y)$	$P(x,y)$ is true for every pair x,y	There is at least one *pair* x,y for which $P(x,y)$ is false
$\forall x \exists y\, P(x,y)$	For every x, there is a y for which $P(x,y)$ is true	There is an x for which $P(x,y)$ is false for every y
$\exists x \forall y\, P(x,y)$	There is an x for which $P(x,y)$ is true for every y	For every x, there is a y for which $P(x,y)$ is false
$\exists x \exists y P(x,y)$	There is at least one pair x,y for which $P(x,y)$ is true	$P(x,y)$ is false for every pair x,y

Next class

- Topic: Quantifiers and Rules of Inference
- Pre-class reading: Chap 1.4-1.5