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Lecture 32: Relations (1)

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Recap Previous Lecture

- Expectation and variance
- Binominal distribution

$$E(X) = \sum_{\text{all } x} x_i \, p(x_i)$$

$$Var(X) = \sum_{\text{all } x} (x_i - \mu)^2 \, p(x_i)$$

$\binom{5}{3}$ ways to arrange 3 heads in 5 trials

Outcome	Probability
THHHT	$(1/2)^3 \, x \, (1/2)^2$
HHHTT	$(1/2)^3 \, x \, (1/2)^2$
TTHHH	$(1/2)^3 \, x \, (1/2)^2$
HTTHH	$(1/2)^3 \, x \, (1/2)^2$
HHTTH	$(1/2)^3 \, x \, (1/2)^2$
HTHHT	$(1/2)^3 \, x \, (1/2)^2$
THTHH	$(1/2)^3 \, x \, (1/2)^2$
HTHTH	$(1/2)^3 \, x \, (1/2)^2$
HHTHT	$(1/2)^3 \, x \, (1/2)^2$
THHTH	$(1/2)^3 \, x \, (1/2)^2$

10 arrangements $x \, (1/2)^3 \, x \, (1/2)^2$

The probability of each unique outcome (note: they are all equal)

C(5,3) = 5!/3!2! = 10

Factorial review: n! = n(n-1)(n-2)…

Outline

- Introduction to Relations
- Properties of Relations
- Combining Relations
- N-ary Relations

Outline

- **Introduction to Relations**
- Properties of Relations
- Combining Relations
- N-ary Relations

Relations

- If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** with their first element taken from A and their second element taken from B.
- Since this is a relation between **two sets**, it is called a **binary relation**.

- **Definition:** Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

- In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and $a\underline{R}b$ to denote that $(a, b) \notin R$.

Relations

- When (a, b) belongs to R, a is said to be **related** to b by R.
- **Example:** Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).
- P = {Carl, Suzanne, Peter, Carla},
- C = {Mercedes, BMW, tricycle}
- D = {(Carl, Mercedes), (Suzanne, Mercedes), (Suzanne, BMW), (Peter, tricycle)}
- This means that Carl drives a Mercedes, Suzanne drives a Mercedes and a BMW, Peter drives a tricycle, and Carla does not drive any of these vehicles.

Functions as Relations

- You might remember that a **function** f from a set A to a set B assigns a unique element of B to each element of A.
- The **graph** of f is the set of ordered pairs (a, b) such that b = f(a).
- Since the graph of f is a subset of $A \times B$, it is a **relation** from A to B.
- Moreover, for each element **a** of A, there is exactly one ordered pair in the graph that has **a** as its first element.

Functions as Relations

- Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph.

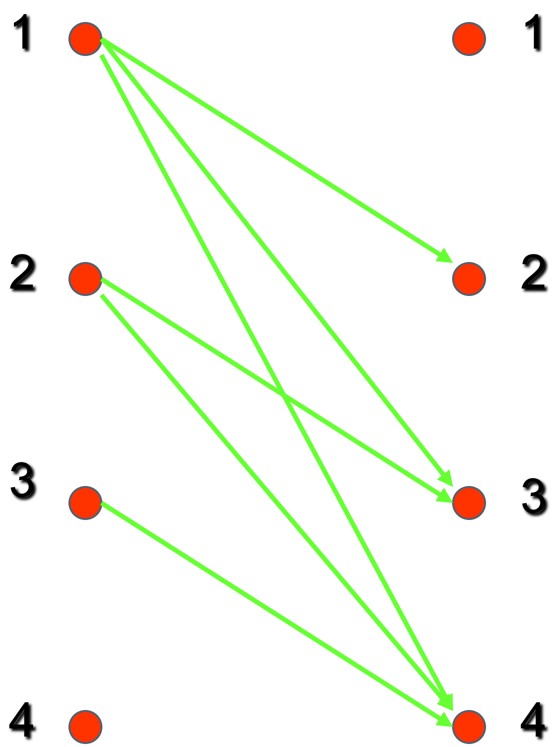
- This is done by assigning to an element $a \in A$ the unique element $b \in B$ such that $(a, b) \in R$.

Relations on a Set

- **Definition:** A relation on the set A is a relation from A to A.
- In other words, a relation on the set A is a subset of $A \times A$.
- **Example:** Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a < b} ?

Relations on a Set

- **Solution:** R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4),}



R	1	2	3	4
1		x	x	x
2			x	x
3				x
4				

Relations on a Set

- **How many different relations can we define on a set A with n elements?**

- A relation on a set A is a subset of $A \times A$.
- How many elements are in $A \times A$?

- There are n^2 elements in $A \times A$, so how many subsets (= relations on A) does $A \times A$ have?

- The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of $A \times A$.

- **Answer:** We can define 2^{n^2} different relations on A.

Outline

- Introduction to Relations
- **Properties of Relations**
- Combining Relations
- N-ary Relations and Its Applications

Properties of Relations

- We will now look at some useful ways to classify relations.
- **Definition:** A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.
- Are the following relations on {1, 2, 3, 4} reflexive?

R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)} No.

R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)} Yes.

R = {(1, 1), (2, 2), (3, 3)} No.

Definition: A relation on a set A is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.

Properties of Relations

- **Definitions:**
- A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- A relation R on a set A is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$.
- A relation R on a set A is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$.

Properties of Relations

- Are the following relations on {1, 2, 3, 4} symmetric, antisymmetric, or asymmetric?

R = {(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)} Symmetric.

R = {(1, 1)} sym. and antisym.

R = {(1, 3), (3, 2), (2, 1)} asym.

R = {(4, 4), (3, 3), (1, 4)} antisym.

Properties of Relations

- **Definition:** A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

- Are the following relations on {1, 2, 3, 4} transitive?

R = {(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)} Yes.

R = {(1, 3), (3, 2), (2, 1)} No.

R = {(2, 4), (4, 3), (2, 3), (4, 1)} No.

Counting Relations

- **Example:** How many different reflexive relations can be defined on a set A containing n elements?

- **Solution:** Relations on R are subsets of $A \times A$, which contains n^2 elements.
- Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.
- A **reflexive** relation, however, **must** contain the n elements (a, a) for every $a \in A$.
- Consequently, we can only choose among $n^2 - n = n(n - 1)$ elements to generate reflexive relations, so there are $2^{n(n-1)}$ of them.

Counting Relations

- Relations are sets, and therefore, we can apply the usual **set operations** to them.
- If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.
- In each case, the result will be **another relation from A to B**.

Counting Relations

- **Definition:** Let R be a relation from a set A to a set B and S a relation from B to a set C.
- The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by **$S \circ R$**.

- In other words, if relation R contains a pair (a, b) and relation S contains a pair (b, c), then $S \circ R$ contains a pair (a, c).

Counting Relations

- **Example:** Let D and S be relations on A = {1, 2, 3, 4}.
- D = {(a, b) | b = 5 - a}   "b equals (5 – a)"
- S = {(a, b) | a < b}   "a is smaller than b"

- D = {(1, 4), (2, 3), (3, 2), (4, 1)}
- S = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}
- S∘D = {(2,4), (3,3), (3,4), (4,2), (4,3), (4,4)}

D maps an element a to the element (5 – a), and afterwards S maps (5 – a) to all elements larger than (5 – a), resulting in **S∘D = {(a,b) | b > 5 – a}** or **S∘D = {(a,b) | a + b > 5}.**

Outline

- Introduction to Relations
- Properties of Relations
- **Combining Relations**
- N-ary Relations

Combining Relations

- We already know that **functions** are just **special cases** of **relations** (namely those that map each element in the domain onto exactly one element in the codomain).

- If we formally convert two functions into relations, that is, write them down as sets of ordered pairs, the composite of these relations will be exactly the same as the composite of the functions (as defined earlier).

Combining Relations

- **Definition:** Let R be a relation on the set A. The powers R^n, n = 1, 2, 3, …, are defined inductively by
- $R^1 = R$
- $R^{n+1} = R^n \circ R$

- In other words:
- $R^n = R \circ R \circ \ldots \circ R$ (n times the letter R)

Combining Relations

- **Theorem:** The relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.
- **Remember the definition of transitivity:**
- **Definition:** A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.
- The composite of R with itself contains exactly these pairs (a, c).
- Therefore, for a transitive relation R, $R \circ R$ does not contain any pairs that are not in R, so $R \circ R \subseteq R$.
- Since $R \circ R$ does not introduce any pairs that are not already in R, it must also be true that $(R \circ R) \circ R \subseteq R$, and so on, so that $R^n \subseteq R$.

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n-ary Relations

- In order to study an interesting application of relations, namely **databases**, we first need to generalize the concept of binary relations to **n-ary relations**.

- **Definition:** Let $A_1, A_2, \ldots, A_n$ be sets. An **n-ary relation** on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.
- The sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation, and n is called its **degree**.

n-ary Relations

- **Example:**
- Let R = {(a, b, c) | a = 2b $\wedge$ b = 2c with a, b, c$\in$**N**}
- What is the degree of R?
- The degree of R is 3, so its elements are triples.
- What are its domains?
- Its domains are all equal to the set of integers.
- Is (2, 4, 8) in R?
- No.
- Is (4, 2, 1) in R?
- Yes.

Next class

- Topic: Relations (2)
- Pre-class reading: Chap 9.3-9.5