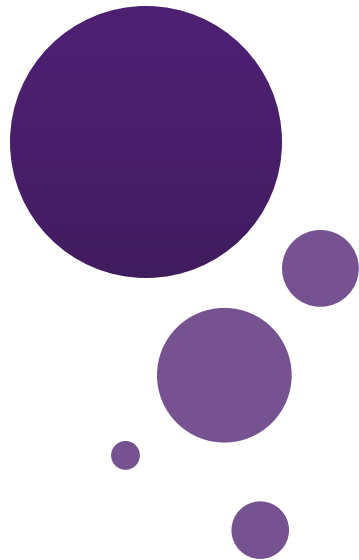




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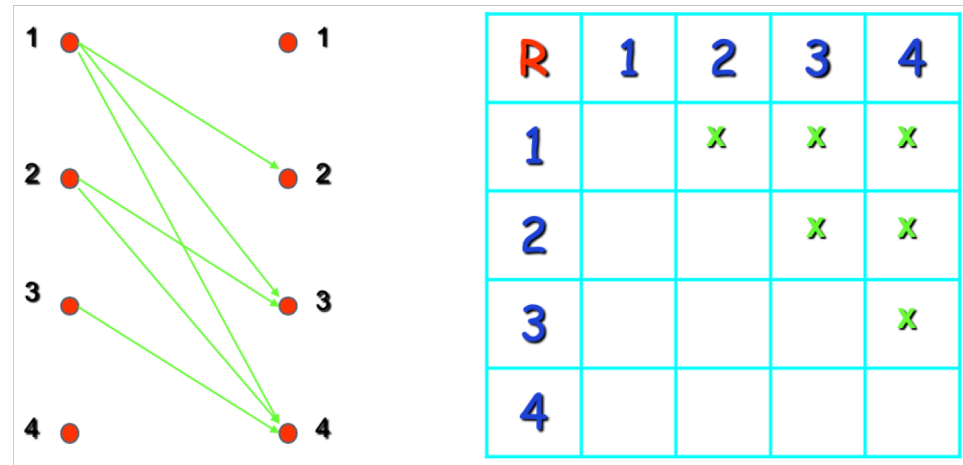
Lecture 33: Relations (2)



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Recap Previous Lecture

- Relations
- Properties of Relations
- Combining Relations
- N-ary Relations



$P = \{\text{Carl, Suzanne, Peter, Carla}\},$
 $C = \{\text{Mercedes, BMW, tricycle}\}$
 $D = \{(\text{Carl, Mercedes}), (\text{Suzanne, Mercedes}),$
 $\quad (\text{Suzanne, BMW}), (\text{Peter, tricycle})\}$

If we have two relations R_1 and R_2 , and both of them are from a set A to a set B , then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.

$$S \circ D$$

$$R^n = R \circ R \circ \dots \circ R$$

$$R \subseteq A_1 \times A_2 \times \dots \times A_n$$

Outline

- Representing Relations Using Matrices
- Representing Relations Using Digraph
- Equivalence Relations
- Equivalence Classes
- Application Example

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- **Representing Relations Using Matrices**
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Representing Relations Using Matrices

- We already know different ways of representing relations. We will now take a closer look at two ways of representation: **Zero-one matrices** and **directed graphs**.

- If R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with

- $m_{ij} = 1$, if $(a_i, b_j) \in R$, and

- $m_{ij} = 0$, if $(a_i, b_j) \notin R$.

- Note that for creating this matrix we first need to list the elements in A and B in a **particular, but arbitrary order**.

Representing Relations Using Matrices

• **Example:** How can we represent the relation $R = \{(2, 1), (3, 1), (3, 2)\}$ as a zero-one matrix?

• **Solution:** The matrix M_R is given by

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Representing Relations Using Matrices

- What do we know about the matrices representing a **relation on a set** (a relation from A to A) ?
- They are **square** matrices.
- What do we know about matrices representing **reflexive** relations?
- All the elements on the **diagonal** of such matrices M_{ref} must be **1s**.

$$M_{ref} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix}$$

Representing Relations Using Matrices

- What do we know about the matrices representing **symmetric relations**?
- These matrices are symmetric, that is, $M_R = (M_R)^t$.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

**symmetric matrix,
symmetric relation.**

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

**non-symmetric matrix,
non-symmetric relation.**

Representing Relations Using Matrices

- The Boolean operations **join** and **meet** (you remember?) can be used to determine the matrices representing the **union** and the **intersection** of two relations, respectively.
- To obtain the **join** of two zero-one matrices, we apply the Boolean “or” function to all corresponding elements in the matrices.
- To obtain the **meet** of two zero-one matrices, we apply the Boolean “and” function to all corresponding elements in the matrices.

Representing Relations Using Matrices

- **Example:** Let the relations R and S be represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R \cup S$ and $R \cap S$?

Solution: These matrices are given by

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Representing Relations Using Matrices

- Do you remember the **Boolean product** of two zero-one matrices?
- Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix.
- Then the **Boolean product** of A and B , denoted by $A \circ B$, is the $m \times n$ matrix with (i, j) th entry $[c_{ij}]$, where
- $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \dots \vee (a_{ik} \wedge b_{kj})$.
- $c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n ; otherwise $c_{ij} = 0$.

Representing Relations Using Matrices

- Let us now assume that the zero-one matrices $M_A = [a_{ij}]$, $M_B = [b_{ij}]$ and $M_C = [c_{ij}]$ represent relations A, B, and C, respectively.
- **Remember:** For $M_C = M_A \circ M_B$ we have:
- $c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n ; otherwise $c_{ij} = 0$.
- In terms of the **relations**, this means that C contains a pair (x_i, z_j) if and only if there is an element y_n such that (x_i, y_n) is in relation A and (y_n, z_j) is in relation B.
- Therefore, $C = B \circ A$ (**composite** of A and B).

Representing Relations Using Matrices

- This gives us the following rule:
- $M_{B \circ A} = M_A \circ M_B$
- In other words, the matrix representing the **composite** of relations A and B is the **Boolean product** of the matrices representing A and B.
- Analogously, we can find matrices representing the **powers of relations**:
- $M_{R^n} = M_R^{[n]}$ (n-th **Boolean power**).

Representing Relations Using Matrices

- **Example:** Find the matrix representing R^2 , where the matrix representing R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution: The matrix for R^2 is given by

$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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- **Representing Relations Using Digraph**
- Equivalence Relations
- Equivalence Classes
- Application Example

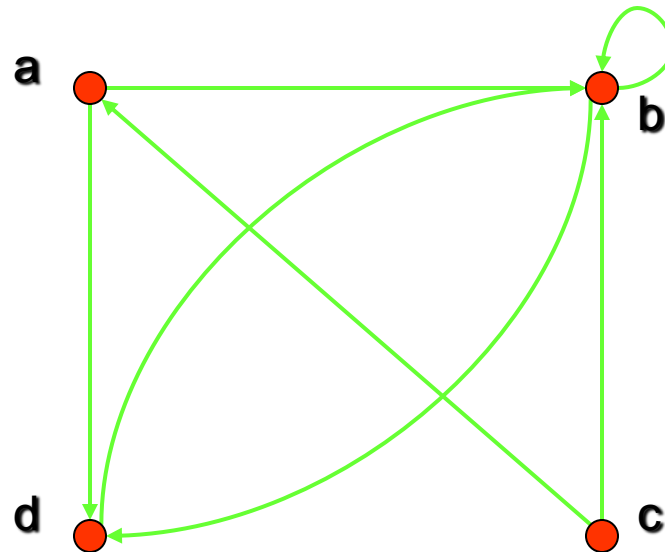
Representing Relations Using Digraphs

- **Definition:** A **directed graph**, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**).
- The vertex a is called the **initial vertex** of the edge (a, b) , and the vertex b is called the **terminal vertex** of this edge.

- We can use arrows to display graphs.

Representing Relations Using Digraphs

- **Example:** Display the digraph with $V = \{a, b, c, d\}$, $E = \{(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)\}$.



An edge of the form (b, b) is called a **loop**.

Representing Relations Using Digraphs

- Obviously, we can represent any relation R on a set A by the digraph with A as its vertices and all pairs $(a, b) \in R$ as its edges.
- Vice versa, any digraph with vertices V and edges E can be represented by a relation on V containing all the pairs in E .
- This **one-to-one correspondence** between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.

Outline

- Representing Relations Using Matrices
- Representing Relations Using Digraph
- **Equivalence Relations**
- Equivalence Classes
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Equivalence Relations

- **Equivalence relations** are used to relate objects that are similar in some way.
- **Definition:** A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.
- Two elements that are related by an equivalence relation R are called **equivalent**.

Equivalence Relations

- Since R is **symmetric**, a is equivalent to b whenever b is equivalent to a .
- Since R is **reflexive**, every element is equivalent to itself.
- Since R is **transitive**, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.
- Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relations

• **Example:** Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x . Is R an equivalence relation?

• **Solution:**

- R is reflexive, because $l(a) = l(a)$ and therefore aRa for any string a .
- R is symmetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if aRb then bRa .
- R is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so aRb and bRc implies aRc .
- **R is an equivalence relation.**

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Equivalence Class

- **Definition:** Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the **equivalence class** of a .
- The equivalence class of a with respect to R is denoted by $[a]_R$.
- When only one relation is under consideration, we will delete the subscript R and write $[a]$ for this equivalence class.
- If $b \in [a]_R$, b is called a **representative** of this equivalence class.

Example

- In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?
- **Solution:** [mouse] is the set of all English words containing five letters.
- For example, 'horse' would be a representative of this equivalence class.

Equivalence Classes

• **Theorem:** Let R be an equivalence relation on a set A . The following statements are equivalent:

- aRb
- $[a] = [b]$
- $[a] \cap [b] \neq \emptyset$
- A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i , $i \in I$, forms a partition of S if and only if
 - $A_i \neq \emptyset$ for $i \in I$
 - $A_i \cap A_j = \emptyset$, if $i \neq j$
 - $\cup_{i \in I} A_i = S$

Example

- Let S be the set $\{u, m, b, r, o, c, k, s\}$.
Do the following collections of sets partition S ?

$\{\{m, o, c, k\}, \{r, u, b, s\}\}$ **yes.**

$\{\{c, o, m, b\}, \{u, s\}, \{r\}\}$ **no (k is missing).**

$\{\{b, r, o, c, k\}, \{m, u, s, t\}\}$ **no (t is not in S).**

$\{\{u, m, b, r, o, c, k, s\}\}$ **yes.**

$\{\emptyset, \{b, o, k\}, \{r, u, m\}, \{c, s\}\}$ **no (\emptyset not allowed).**

Equivalence Classes

- **Theorem:** Let R be an equivalence relation on a set S . Then the **equivalence classes** of R form a **partition** of S . Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S , there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.

Example

- Let R be the relation $\{(a, b) \mid a \equiv b \pmod{3}\}$ on the set of integers. Is R an equivalence relation?
- **Yes, R is reflexive, symmetric, and transitive.**
- What are the equivalence classes of R ?
- $\{\{\dots, -6, -3, 0, 3, 6, \dots\},$
 $\{\dots, -5, -2, 1, 4, 7, \dots\},$
 $\{\dots, -4, -1, 2, 5, 8, \dots\}\}$

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Databases and Relations

- Consider a relational database of students, whose records are represented as 4-tuples with the fields **Student Name**, **ID Number**, **Major**, and **GPA**:
- $R = \{(Ackermann, 231455, CS, 3.88), (Adams, 888323, Physics, 3.45), (Chou, 102147, CS, 3.79), (Goodfriend, 453876, Math, 3.45), (Rao, 678543, Math, 3.90), (Stevens, 786576, Psych, 2.99)\}$
- Relations that represent databases are also called **tables**, since they are often displayed as tables.

Databases and Relations

- A domain of an n-ary relation is called a **primary key** if the n-tuples are uniquely determined by their values from this domain.
- This means that no two records have the same value from the same primary key.
- In our example, which of the fields **Student Name**, **ID Number**, **Major**, and **GPA** are primary keys?
- **Student Name** and **ID Number** are primary keys, because no two students have identical values in these fields.
- In a real student database, only **ID Number** would be a primary key.

Databases and Relations

- We can apply a variety of **operations** on n-ary relations to form new relations.
- **Definition:** The **projection** P_{i_1, i_2, \dots, i_m} maps the n-tuple (a_1, a_2, \dots, a_n) to the m-tuple $(a_{i_1}, a_{i_2}, \dots, a_{i_m})$, where $m \leq n$.
- In other words, a projection P_{i_1, i_2, \dots, i_m} keeps the m components $a_{i_1}, a_{i_2}, \dots, a_{i_m}$ of an n-tuple and deletes its $(n - m)$ other components.
- **Example:** What is the result when we apply the projection $P_{2,4}$ to the student record (Stevens, 786576, Psych, 2.99) ?
- **Solution:** It is the pair (786576, 2.99).

Databases and Relations

- We can use the **join** operation to combine two tables into one if they share some identical fields.

- **Definition:** Let R be a relation of degree m and S a relation of degree n . The **join** $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$ -tuples

$(a_1, a_2, \dots, a_{m-p}, c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_{n-p})$,

where the m -tuple $(a_1, a_2, \dots, a_{m-p}, c_1, c_2, \dots, c_p)$ belongs to R and the n -tuple $(c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_{n-p})$ belongs to S .

Databases and Relations

• **Example:** What is $J_1(Y, R)$, where Y contains the fields **Student Name** and **Year of Birth**,

• $Y = \{(1978, \text{Ackermann}),$
 $(1972, \text{Adams}),$
 $(1917, \text{Chou}),$
 $(1984, \text{Goodfriend}),$
 $(1982, \text{Rao}),$
 $(1970, \text{Stevens})\},$

$R = \{(\text{Ackermann}, 231455, \text{CS}, 3.88),$
 $(\text{Adams}, 888323, \text{Physics}, 3.45),$
 $(\text{Chou}, 102147, \text{CS}, 3.79),$
 $(\text{Goodfriend}, 453876, \text{Math}, 3.45),$
 $(\text{Rao}, 678543, \text{Math}, 3.90),$
 $(\text{Stevens}, 786576, \text{Psych}, 2.99)\}$

• and R contains the student records as defined before ?

Databases and Relations

- **Solution:** The resulting relation is:
 - $\{(1978, \text{Ackermann}, 231455, \text{CS}, 3.88), (1972, \text{Adams}, 888323, \text{Physics}, 3.45), (1917, \text{Chou}, 102147, \text{CS}, 3.79), (1984, \text{Goodfriend}, 453876, \text{Math}, 3.45), (1982, \text{Rao}, 678543, \text{Math}, 3.90), (1970, \text{Stevens}, 786576, \text{Psych}, 2.99)\}$
- Since Y has two fields and R has four, the relation $J_1(Y, R)$ has $2 + 4 - 1 = 5$ fields.

Next class

- Topic: Graph Theory (1)
- Pre-class reading: Chap 10.1-10.2

