Lecture 6: Introduction to Sets

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Recap Previous Lecture

- Proofs
  - Trivial proof (without using the premise)
  - Vacuous proof (the premise p is false)
  - Direct proof ($p \rightarrow q$)
  - Indirect / Contrapositive proof ($\neg q \rightarrow \neg p$)
  - Proof by contradiction ($\neg p \rightarrow (r \wedge \neg r)$, where r is a known result)
  - Proof by cases (break the statements down into cases and prove each one separately)
  - Existence Proofs ($\neg \exists x P(x)$)
    - constructive existence proof (a specific, concrete example)
    - nonconstructive existence proof ($\neg \exists x P(x) \equiv \forall x \neg P(x)$ with contradiction)
  - Uniqueness Proofs ($\exists x (P(x) \wedge (\forall y (x \neq y \rightarrow \neg P(y)))$)

Counter Examples

- Sometimes you are asked to <u>disprove</u> a statement
- In such a situation you are actually trying to prove the <u>negation</u> of the statement
- With statements of the form $\forall x\ P(x)$, it suffices to give a <u>counter example</u>
  - because the existence of an element x for which $\neg P(x)$ holds proves that $\exists x\ \neg P(x)$
  - which is the negation of $\forall x\ P(x)$

Counter Examples

- Example: Disprove n^2+n+1 is a prime number for all $n \geq 1$
- A simple counter example is n=4.
- In fact: for n=4, we have

$$n^2+n+1 = 4^2+4+1$$
$$= 16+4+1$$
$$= 21 = 3 \times 7, \text{ which is clearly not prime} \qquad \text{QED}$$

Counter Examples: A Word of Caution

- No matter how many examples you give, you can never prove a theorem by giving examples (unless the universe of discourse is finite—why?—which is in called an exhaustive proof)
- Counter examples can only be used to disprove universally quantified statements
- Do not give a proof by simply giving an example

Proof Strategies

- Forward and backward reasoning
- If there were a single strategy that always worked for proofs, mathematics would be easy
- The best advice we can give you:
  - Beware of fallacies and circular arguments (i.e., begging the question)
  - Don't take things for granted. Try proving assertions first before you can take/use them as facts
  - Don't peek at proofs. Try proving something for yourself before looking at the proof
  - If you peeked, challenge yourself to reproduce the proof later on.. w/o peeking again
  - The best way to improve your proof skills is PRACTICE.

Outline

- Definitions: Set, Element
- Terminology and Notation
- Proving Equivalences
- Power Set
- Tuples
- Cartesian Product
- Quantifiers

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- **Definitions: Set, Element**
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Introduction (1)

- We have already implicitly dealt with sets
  - Integers ($\mathbb{Z}$), rationals ($\mathbb{Q}$), naturals ($\mathbb{N}$), reals ($\mathbb{R}$), etc.
- We will develop more fully
  - The definitions of sets
  - The properties of sets
  - The operations on sets
- **Definition**: A set is an unordered collection of (unique) objects
- Sets are fundamental discrete structures and for the basis of more complex discrete structures like graphs

Introduction (2)

- **Definition**: The objects in a set are called <u>elements</u> or <u>members</u> of a set. A set is said to contain its elements
- Notation, for a set A:
  - $x \in A$: x is an element of A \$\in\$
  - $x \notin A$: x is not an element of A \$\notin\$

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Terminology (1)

- **Definition**: Two sets, A and B, are equal is they contain the same elements. We write A=B.
- Example:
  - {2,3,5,7}={3,2,7,5}, because a set is unordered
  - Also, {2,3,5,7}={2,2,3,5,3,7} because a set contains unique elements
  - However, $\{2,3,5,7\} \neq \{2,3\}$ $\neq$

A set is a collection of well defined and distinct objects.

Terminology (2)

- A multi-set is a set where you specify the number of occurrences of each element: $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ is a set where
  - m_1 occurs a_1 times
  - m_2 occurs a_2 times
  - …
  - m_r occurs a_r times
- In Databases, we distinguish
  - A set: elements cannot be repeated
  - A bag: elements can be repeated

Terminology (3)

- The **set-builder** notation

$$O=\{ x \mid (x \in \mathbb{Z}) \wedge (x=2k) \text{ for some } k \in Z\}$$

reads: O is the set that contains all x such that x is an integer and x is even

- A set is defined in **intension** when you give its set-builder notation

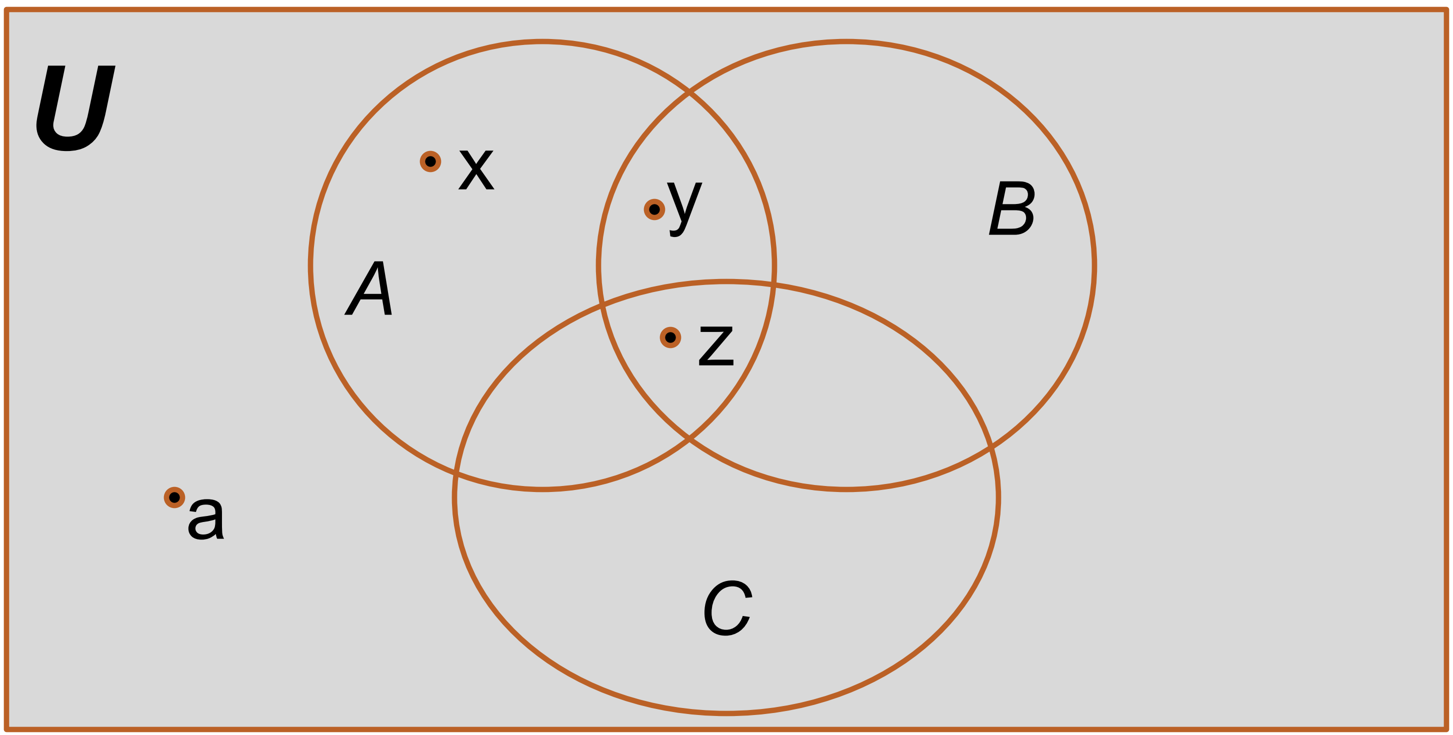
$$O=\{ x \mid (x \in \mathbb{Z}) \wedge (0 \leq x \leq 8) \wedge (x=2k) \text{ for some } k \in Z \}$$

- A set is defined in **extension** when you enumerate all the elements:

$$O=\{0,2,4,6,8\}$$

Venn Diagram: Example

- A set can be represented graphically using a Venn Diagram



More Terminology and Notation (1)

- A set that has no elements is called the **empty set** or **null set** and is denoted $\emptyset$ $\emptyset$
- A set that has one element is called a **singleton set**.
  - For example: {a}, with brackets, is a singleton set
  - a, without brackets, is an element of the set {a}
- Note the subtlety in $\emptyset \neq \{\emptyset\}$
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set

More Terminology and Notation (2)

- **Definition**: A is said to be a **subset** of B, and we write $A \subseteq B$, if and only if every element of A is also an element of B $\subseteq$ $\backslash$subseteq$
- That is, we have the equivalence:

$$A \subseteq B \Leftrightarrow \forall x \, (x \in A \Rightarrow x \in B)$$

More Terminology and Notation (3)

- **Theorem**: For any set S *Theorem 1, page 115*
  - $\varnothing \subseteq S$ and
  - $S \subseteq S$
- The proof is in the book, an excellent example of a vacuous proof

More Terminology and Notation (4)

- **Definition**: A set A that is a subset of a set B is called a **proper subset** if $A \neq B$.
- That is there is an element $x \in B$ such that $x \notin A$
- We write: $A \subset B$, $A \subsetneq B$
- In LaTex: $\subset$, $\subsetneq$

More Terminology and Notation (5)

- Sets can be elements of other sets
- Examples
  - $S_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, c\}$
  - $S_2 = \{\{1\}, \{2,4,8\}, \{3\}, \{6\}, 4, 5, 6\}$

More Terminology and Notation (6)

- **Definition**: If there are exactly n distinct elements in a set S, with n a nonnegative integer, we say that:
  - S is a finite set, and
  - The cardinality of S is n. Notation: $|S| = n$.
- **Definition**: A set that is not finite is said to be infinite

More Terminology and Notation (7)

- Examples
  - Let B = {x | ($x \leq 100$) $\wedge$ (x is prime)}, the cardinality of B is |B|=25 because there are 25 primes less than or equal to 100.
  - The cardinality of the empty set is $|\emptyset|=0$
  - The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are all infinite

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- Definitions: Set, Element
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- **Proving Equivalences**
- Power Set
- Tuples
- Cartesian Product
- Quantifiers

Proving Equivalence (1)

- You may be asked to show that a set is
  - a subset of,
  - proper subset of, or
  - equal to another set.
- To prove that A is a subset of B, use the equivalence discussed earlier $A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$
  - To prove that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element x, $x \in A$ implies that x is also in B.
  - Any proof method can be used.
- To prove that A is a proper subset of B, you must prove
  - A is a subset of B **and**
  - $\exists x\ (x \in B) \wedge (x \notin A)$

Proving Equivalence (2)

- Finally to show that two sets are equal, it is sufficient to show independently (much like a biconditional) that
  - $A \subseteq B$ and
  - $B \subseteq A$
- Logically speaking, you must show the following quantified statements:

$$(\forall x\,(x \in A \Rightarrow x \in B)) \land (\forall x\,(x \in B \Rightarrow x \in A))$$

we will see an example later..

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Power Set (1)

- **Definition**: The power set of a set S, denoted P(S), is the set of all subsets of S.
- Examples
  - Let A={a,b,c}, P(A)={∅,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}
  - Let A={{a,b},c}, P(A)={∅,{{a,b}},{c},{{a,b},c}}
- Note: the empty set ∅ and the set itself are always elements of the power set. This fact follows from Theorem 1 (Rosen, page 115).

Power Set (2)

- The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set
- **Fact**: Let S be a set such that |S|=n, then

$$|P(S)| = 2^n$$

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Tuples (1)

- Sometimes we need to consider ordered collections of objects
- **Definition**: The ordered n-tuple $(a_1,a_2,\ldots,a_n)$ is the ordered collection with the element a_i being the i-th element for $i=1,2,\ldots,n$
- Two ordered n-tuples $(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$ are equal iff for every $i=1,2,\ldots,n$ we have $a_i=b_i$ $(a_1,a_2,\ldots,a_n)$
- A 2-tuple (n=2) is called an ordered pair

Cartesian Product (1)

- **Definition**: Let A and B be two sets. The Cartesian product of A and B, denoted AxB, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$

$$A \times B = \{ (a,b) \mid (a \in A) \wedge (b \in B) \}$$

- The Cartesian product is also known as the cross product
- **Definition**: A subset of a Cartesian product, $R \subseteq A \times B$ is called a relation. We will talk more about relations in the next set of slides
- Note: $A \times B \neq B \times A$ unless $A=\emptyset$ or $B=\emptyset$ or $A=B$. Find a counter example to prove this.

Cartesian Product (2)

- Cartesian Products can be generalized for any n-tuple
- **Definition**: The Cartesian product of n sets, $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is

$$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n \}$$

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Notation with Quantifiers

- Whenever we wrote $\exists x P(x)$ or $\forall x P(x)$, we specified the universe of discourse using explicit English language
- Now we can simplify things using set notation!
- Example
  - $\forall\ x \in \mathbb{R}\ (x^2 \geq 0)$
  - $\exists\ x \in \mathbb{Z}\ (x^2 = 1)$
  - Also mixing quantifiers:

$$\forall a,b,c \in \mathbb{R}\ \exists\ x \in \mathbb{C}\ (ax^2+bx+c=0)$$

Next class

- Topic: Set Operations and Introduction to Functions
- Pre-class reading: Chap 2.2-2.3