

University at Albany, SUNY

College of Engineering and Applied Sciences, Computer Science

ICSI-521: Discrete Mathematics with Applications Spring 2019

Homework Set 5

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Assigned Date: Mar 28, 2019 (Thursday).

Due Date: Apr 11, 2018 (Thursday), 11:59 PM.

Collaboration Policy. Homeworks will be done individually: each student must hand in their own answers. Use of partial or entire solutions obtained from others or online is strictly prohibited.

Late Policy. If urgent or unusual circumstances prohibit you from submitting a homework assignment in time, please e-mail the instructor explaining the situation to get exempt from late penalty. Otherwise, any late submissions without consent from the instructor will result in exponential penalty – late for one day loses 25%, two days loses 50%, and so on and so forth. **Those submissions ≥ 3 hours after the deadline will be considered as “late submission” with no exemption.**

Submission Format. Electronic submission as a zip file including a PDF file and code files to blackboard is mandatory.

- You can write your solution in Word and save it as a PDF file.
- You also can write it on any physical papers and scan them to a PDF file.
- If you don't have condition to scan, you still can take pictures by your smart phone and convert images to a PDF file by the online tool (<https://imagetopdf.com>).
- If you have multiple PDF files, please combine them to a PDF file by the online tool (<https://www.pdfmerge.com>) or (https://www.ilovepdf.com/merge_pdf).

Problem 1: Discrete Probabilities (10 points)

(a) [3 points] What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?

(b) [4 points] What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

(c) [5 points] Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.

(d) [3 points] Suppose that E and F are events in a sample space and $p(E) = 2/3$, $p(F) = 3/4$, and $p(F|E) = 5/8$. Find $p(E|F)$.

(e) [5 points] Suppose that E , F_1 , F_2 , and F_3 are events from a sample space S and that F_1 , F_2 , and F_3 are pairwise disjoint and their union is S . Find $p(F_2|E)$ if $p(E|F_1) = 2/7$, $p(E|F_2) = 3/8$, $p(E|F_3) = 1/2$, $p(F_1) = 1/6$, $p(F_2) = 1/2$, and $p(F_3) = 1/3$.

Problem 2: Bayes Rule (10 points) Suppose that 8% of the patients tested in a clinic are infected with HIV. Furthermore, suppose that when a blood test for HIV is given, 98% of the patients infected with HIV test positive and that 3% of the patients not infected with HIV test positive. What is the probability that

- (a) a patient testing positive for HIV with this test is infected with it?
- (b) a patient testing positive for HIV with this test is not infected with it?
- (c) a patient testing negative for HIV with this test is infected with it?
- (d) a patient testing negative for HIV with this test is not infected with it?

Problem 3: Expected Values and Variance (6 points)

(a) [4 points] What is the expected number of heads that come up when a fair coin is flipped five times?

(b) [4 points] What is the expected number of times a 6 appears when a fair die is rolled 10 times?

(c) [8 points] Let X be the number appearing on the first die when two fair dice are rolled and let Y be the sum of the numbers appearing on the two dice. Show that $E(X)E(Y) \neq E(XY)$.

Problem 4: Properties of Relations (14 points) Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

- (a) $x \neq y$.
- (b) $xy \geq 1$.
- (c) $x = y + 1$ or $x = y - 1$.
- (d) $x \equiv y \pmod{7}$.
- (e) x is a multiple of y .
- (f) x and y are both negative or both nonnegative.
- (g) $x \geq y^2$.

Problem 5: Relations (20 points)

(a) [4 points] Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$.

(b) [3 points] Represent the relation $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$ on the set $S = 1, 2, 3$ with a matrix (with the elements of this set listed in increasing order).

(c) [3 points] Represent the relation $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$ on the set $S = 1, 2, 3$ with a directed graph.

(d) [10 points] Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), (5, 2)$, and $(5, 4)$. Find R^2 , R^3 , R^4 , R^5 , and R^6 .

Problem 6: Computer Projects (40 points)

Write a program with any programming language you like to solve and answer the following problems. Please keep in mind that you should provide details to run your program and testing cases with necessary descriptions, as well as the complexity analysis, to make a solid solution.

(a) [15 points] **Different Ways to Add Parentheses:** Given a string of numbers and operators, return all possible results from computing all the different possible ways to group numbers and operators. The valid operators are $+$, $-$ and \star .

Example 1

Input: "2-1-1"

Output: [0, 2]

Explanation:

$$((2 - 1) - 1) = 0$$

$$(2 - (1 - 1)) = 2$$

Example 2

Input: "2 * 3 - 4 * 5"

Output: [-34, -14, -10, -10, 10]

Explanation:

$$(2 * (3 - (4 * 5))) = -34$$

$$((2 * 3) - (4 * 5)) = -14$$

$$((2 * (3 - 4)) * 5) = -10$$

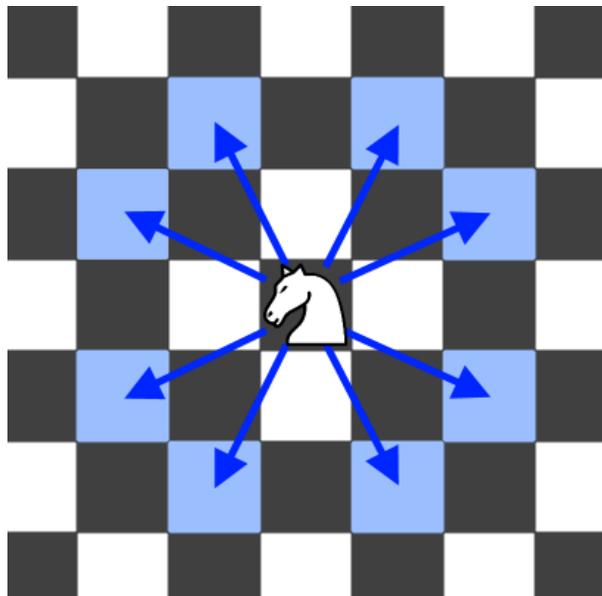
$$(2 * ((3 - 4) * 5)) = -10$$

$$(((2 * 3) - 4) * 5) = 10$$

(b) [10 points] **Knight Probability in Chessboard:** On an $N \times N$ chessboard, a knight starts at the r -th row and c -th column and attempts to make exactly K moves. The rows and columns are 0 indexed, so the top-left square is $(0, 0)$, and the bottom-right square is $(N - 1, N - 1)$.

A chess knight has 8 possible moves it can make, as illustrated below. Each move is two squares in a cardinal direction, then one square in an orthogonal

direction.



Each time the knight is to move, it chooses one of eight possible moves uniformly at random (even if the piece would go off the chessboard) and moves there.

The knight continues moving until it has made exactly K moves or has moved off the chessboard. Return the probability that the knight remains on the board after it has stopped moving.

Note: (1) N will be between 1 and 25. (2) K will be between 0 and 100. (3) The knight always initially starts on the board.

Example

Input: 3, 2, 0, 0

Output: 0.0625

Explanation: There are two moves (to (1,2), (2,1)) that will keep the knight on the board. From each of those positions, there are also two moves that will keep the knight on the board.

The total probability the knight stays on the board is 0.0625.

[Optional Problem] (20 points)

(a) [6 points] What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

(b) [6 points] Use mathematical induction to prove that if E_1, E_2, \dots, E_n is a sequence of n pairwise disjoint events in a sample space S , where n is a positive integer, then $p(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n p(E_i)$.

(c) [8 points] Let R be a reflexive relation on a set A . Show that R^n is reflexive for all positive integers n . [**Hint:** *use mathematic induction to prove.*]