University at Albany, SUNY

College of Engineering and Applied Sciences, Computer Science

ISEN/ISCI-210: Discrete Structures

Fall 2018

Homework Set 4

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Assigned Date: Nov 5, 2018 (Monday).
Due Date: Nov 14, 2018 (Wednesday).
Collaboration Policy. Homeworks will be done individually: each student must hand in their own answers. Use of partial or entire solutions obtained from others or online is strictly prohibited.
Late Policy. No late submissions will be allowed without consent from the instructor. If urgent or unusual circumstances prohibit you from submitting a homework assignment in time, please e-mail me explaining the situation. ***Those submissions $\geq$ 12 hours after the deadline will be considered as "late submission".***
Submission Format. Electronic submission of a PDF file is mandatory.

Problem 1: Mathematic Induction (30 points) Let $P(n)$ be the statement that $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.

(a) [1 points] What is the statement $P(1)$?.

(b) [1 points] Show that $P(1)$ is true, completing the basis step of the proof.

(c) [2 points] What is the inductive hypothesis?

(d) [2 points] What do you need to prove in the inductive step?

(e) [2 points] Complete the inductive step, identifying where you use the inductive hypothesis.

(f) [2 points] Explain why these steps show that this formula is true whenever n is a positive integer.

(g) [10 points] Redefine $P(n)$ to prove that $1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer.

(h) [10 points] Redefine $P(n)$ to prove that $2^n > n^2$ if n is an integer greater than 4.

Problem 2: Strong Induction and Well-Ordering Property (20 points)

(a) The Fibonacci sequence is defined by $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. with starting values $F_0 = F_1 = 1$. Show that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^2 - \left(\frac{1-\sqrt{5}}{2} \right)^2 \right] \tag{1}$$

(b) Prove the Archimedian property which states that if a and b are positive integers, then there exists some positive integer n such that $na \geq b$.

Problem 3: Recursive Definitions and Structural Induction (8 points)
Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$.

Recursive step: If $(a,b) \in S$, then $(a+2, b+3) \in S$ and $(a+3, b+2) \in S$.

(a) List the elements of S produced by the first five applications of the recursive definition.

(b) Show that $5|a+b$ when $(a,b) \in S$.

Problem 4: Pigeonhole Principle (12 points) Use the pigeonhole principle to prove the following statements.

(a) Show that if there are 30 students in a class, then at least two have last names that begin with the same letter

(b) Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior. Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.

(c) What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Problem 5: Permutations and Combinations (20 points)

(a) Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

(b) Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

(c) How many bit strings contain exactly eight 0s and ten 1s if every 0 must be immediately followed by a 1?

(d) How many bit strings contain exactly five 0s and fourteen 1s if every 0 must be immediately followed by two 1s?

(e) How many bit strings of length 10 contain at least three 1s and at least three 0s?

Problem 6: Binomial Coefficients and Identities (10 points)

(a) [3 points] Find the expansion of $(x+y)^5$ using using the binomial theorem.

(b) [3 points] What is the coefficient of x^9 in $(2-x)^{19}$?

(c) [4 points] The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \leq k \leq 10$, is:

1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

[Optional] Extra Points (20 points)

(a) [5 points] Suppose that k and n are integers with $1 \leq k < n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} \tag{2}$$

which relates terms in Pascal's triangle that form a hexagon.

A **circular r-permutation** of n people is a seating of r of thesen people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

(b) [3 points] Find the number of circular 3-permutations of 5 people.

(c) [6 points] Find a formula for the number of circular r-permutations of n people.

(d) [6 points] Find a formula for the number of ways to seatr of n people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.