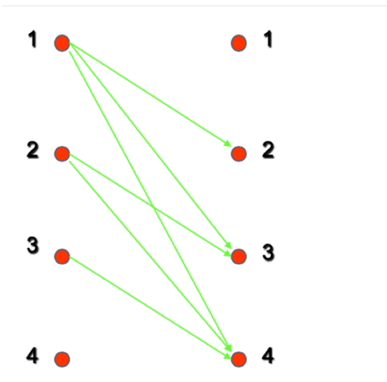
UNIVERSITY AT ALBANY
State University of New York

Lecture 11: Graph Theory

Dr. Chengjiang Long
Computer Vision Researcher at Kitware Inc.
Adjunct Professor at SUNY at Albany.
Email: **clong2@albany.edu**

Recap Previous Lecture

- Relations and its Representation
- Equivalence Relations and Equivalence Classes

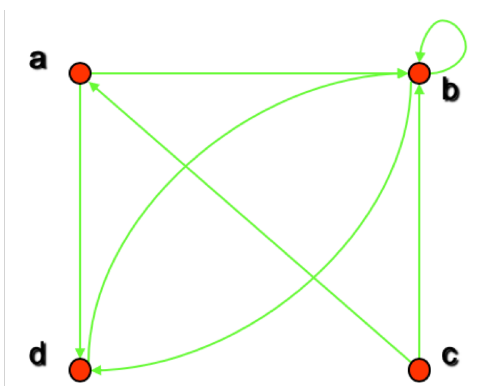


R	1	2	3	4
1		x	x	x
2			x	x
3				x
4				

Let R be the relation {(a, b) | a ≡ b (mod 3)} on the set of integers. Is R an equivalence relation?

- **Yes, R is reflexive, symmetric, and transitive.**
- What are the equivalence classes of R ?
- {{…, -6, -3, 0, 3, 6, …},
  {…, -5, -2, 1, 4, 7, …},
  {…, -4, -1, 2, 5, 8, …}}

V = {a, b, c, d}, E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



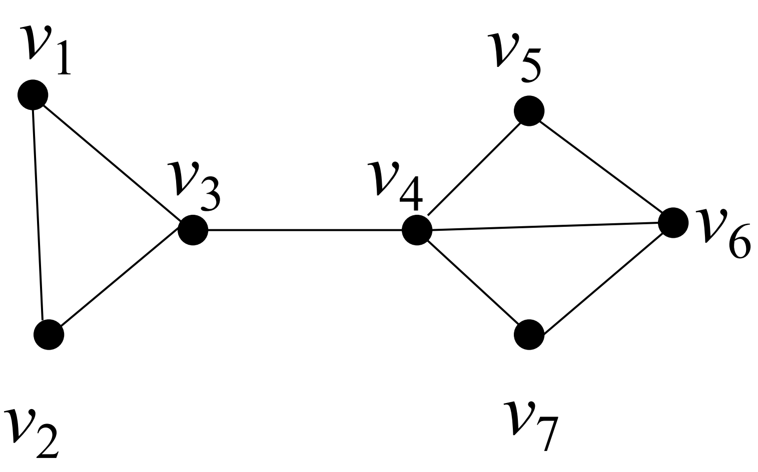
Outline

- Graph and Its Representations
- Euler and Hamiltonian Paths and Shortest-Path Problems
- Planar Graphs and Graph Coloring
- Applications of Graph

Graph and Its Representations

Introduction to Graphs

Def 1. A graph ***G* = (*V*, *E*)** consists of ***V***, a nonempty set of vertices (or nodes), and ***E***, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.



$G=(V, E)$, where

$V=\{v_1,v_2,\ldots, v_7\}$

$E=\{\{v_1,v_2\}, \{v_1,v_3\}, \{v_2,v_3\}$
$\{v_3,v_4\}, \{v_4,v_5\}, \{v_4,v_6\}$
$\{v_4,v_7\}, \{v_5,v_6\}, \{v_6,v_7\}\}$

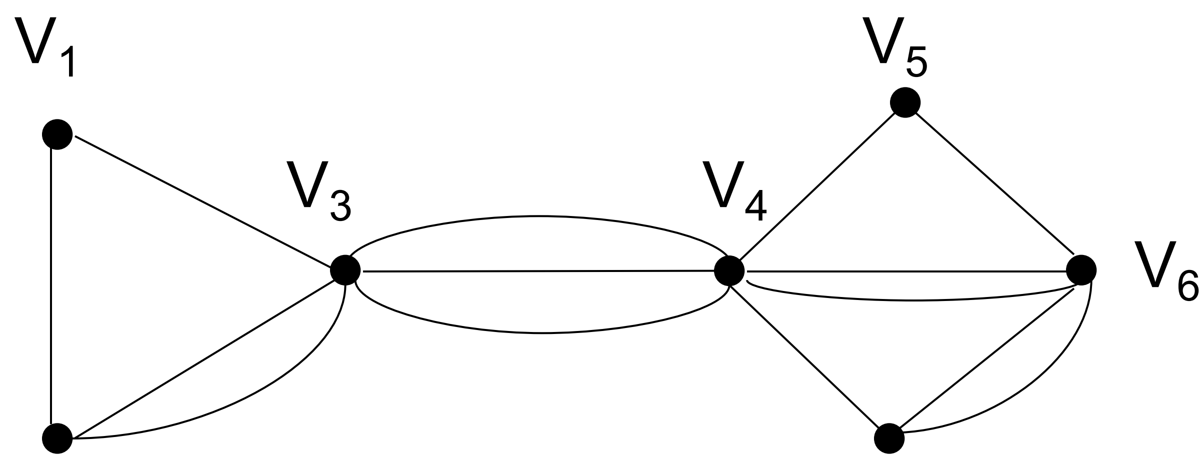
Introduction to Graphs

- A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Multigraph:

simple graph + multiple edges (multiedges)

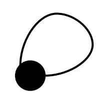
(Between two points to allow multiple edges)

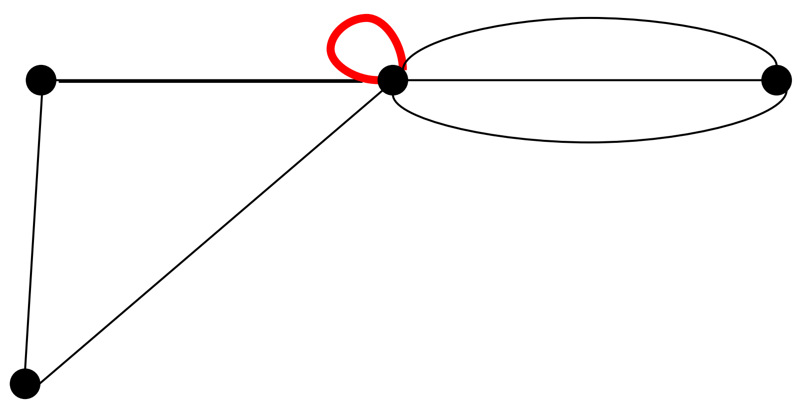


Introduction to Graphs

Def. Pseudograph:

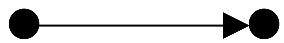
simple graph + multiedge
+ loop

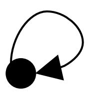
(a loop: )

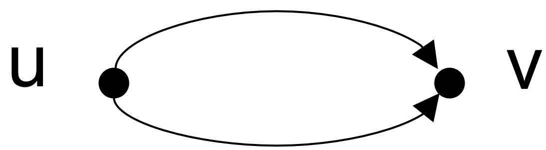


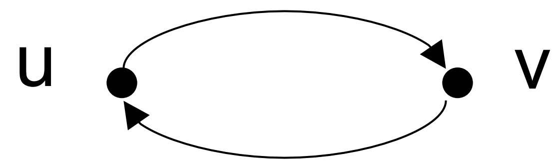
Introduction to Graphs

Def 2. Directed graph (digraph):
simple graph with each edge directed



Note:  is allowed in a directed graph





The two edges (u,v),(u,v) are multiedges.

The two edges (u,v), (v,u) are not multiedges.

Def. Directed multigraph: digraph+multiedges

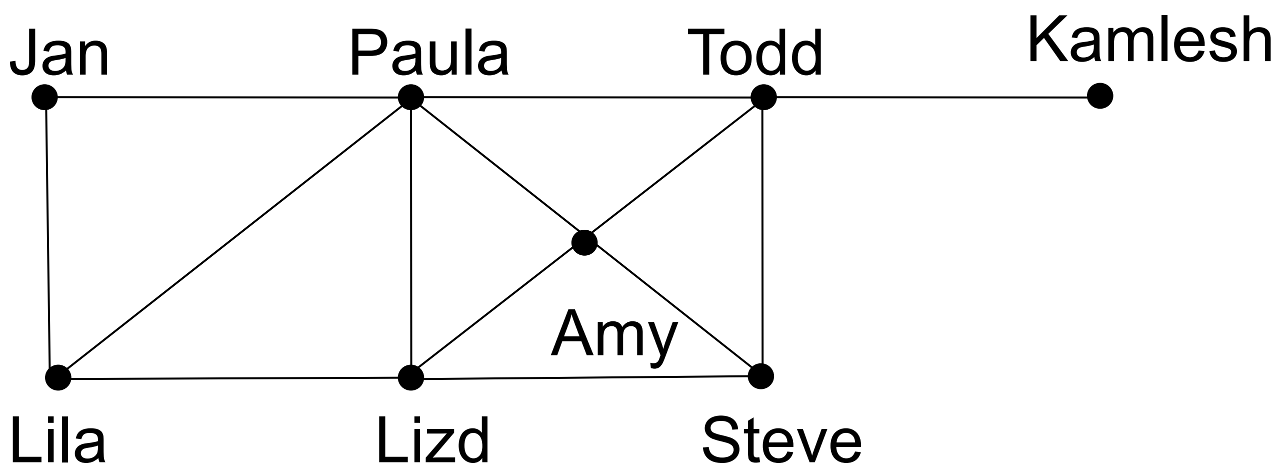
Introduction to Graphs

Table 1. Graph Terminology

Type	Edges	Multiple Edges	Loops
(simple) graph	undirected edge: {u,v}	×	×
Multigraph		✓	×
Pseudograph		✓	✓
Directed graph	directed edge: (u,v)	×	✓
Directed multigraph		✓	✓

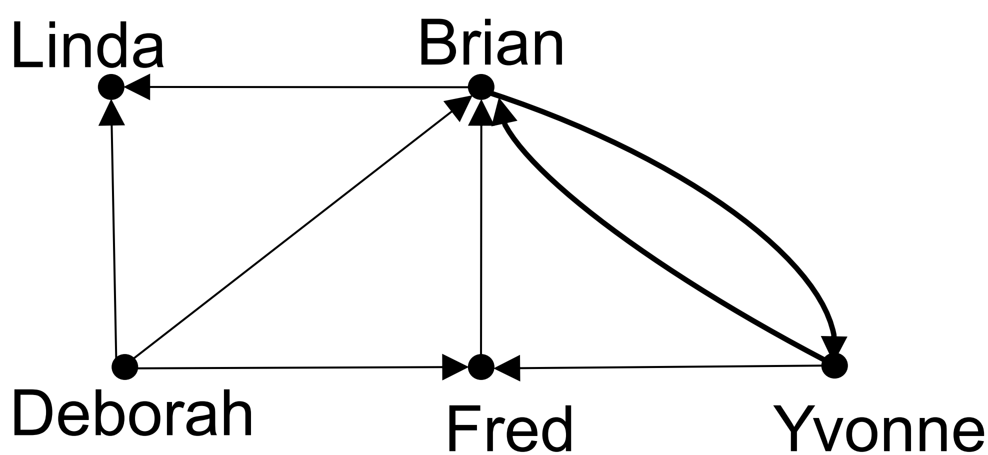
Graph Models: Acquaintance graph

- We can use a simple graph to represent whether two people know each other. Each person is represented by a vertex. An undirected edge is used to connect two people when these people know each other.



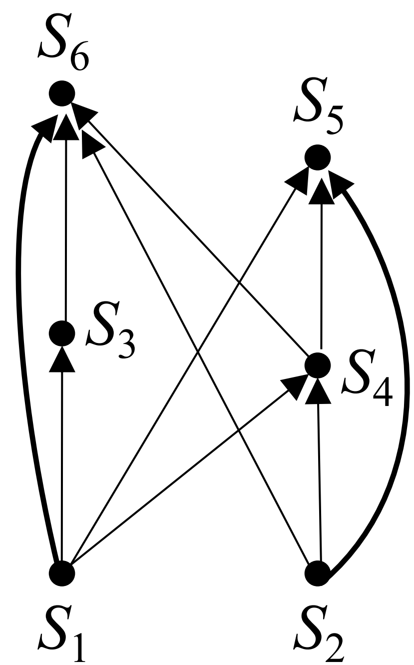
Graph Models: Influence graph

- In studies of group behavior it is observed that certain people can influence the thinking of others.
  Simple digraph $\Rightarrow$ Each person of the group is represented by a vertex. There is a directed edge from vertex a to vertex b when the person a influences the person b.



Graph Models: Precedence graph and concurrent processing

- Computer programs can be executed more rapidly by executing certain statements concurrently. It is important not to execute a statement that requires results of statements not yet executed.
- Simple digraph ⇒ Each statement is represented by a vertex, and there is an edge from a to b if the statement of b cannot be executed before the statement of a.



S_1: a:=0
S_2: b:=1
S_3: c:=a+1
S_4: d:=b+a
S_5: e:=d+1
S_6: e:=c+d

Graph Terminology

Def 1. Two vertices u and v in a undirected graph G are called adjacent (or neighbors) in G if $\{u, v\}$ is an edge of G.

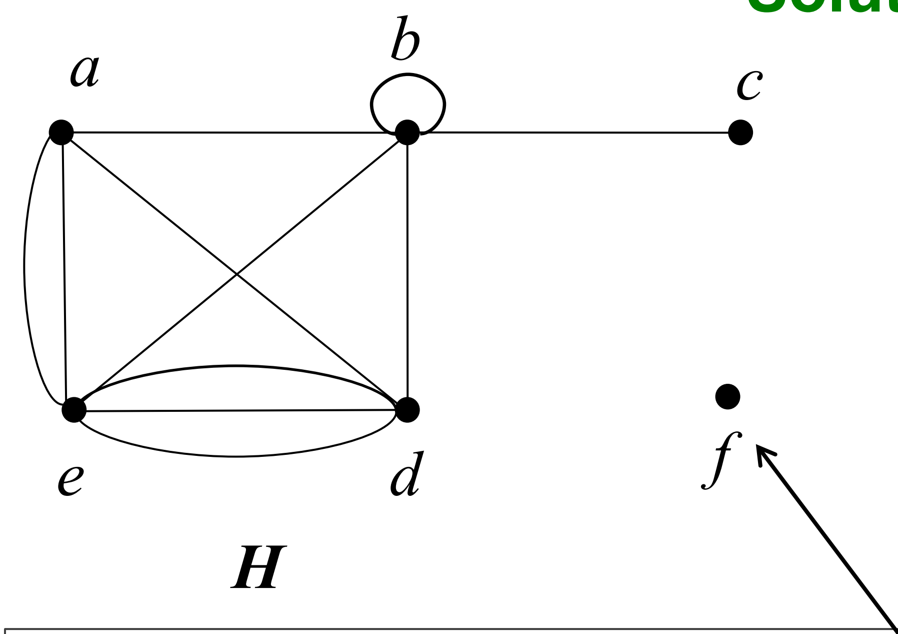
Note : adjacent: a vertex connected to a vertex
incident: a vertex connected to an edge

Def 2. The degree of a vertex v, denoted by $\deg(v)$, in an undirected graph is the number of edges incident with it.

(Note : A loop adds 2 to the degree.)

Example

- What are the degrees of the vertices in the graph H ?



Solution :

$\deg(a)=4$

$\deg(b)=6$

$\deg(c)=1$

$\deg(d)=5$

$\deg(e)=6$

$\deg(f)=0$

Def. A vertex of degree 0 is called isolated.

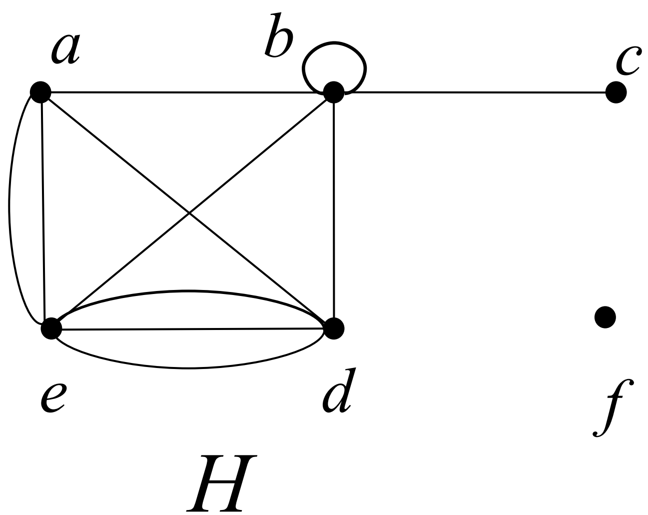
Def. A vertex is **pendant if and only if** it has degree one.

The Handshaking Theorem

- Let $G = (V, E)$ be an undirected graph with e edges (i.e., $|E| = e$). Then

$$\sum_{v \in V} \deg(v) = 2e$$

Example: The graph H has 11 edges, and



$$\sum_{v \in V} \deg(v) = 22$$

Even number of odd degree

- An undirected graph $G = (V, E)$ has an even number of vertices of odd degree.

Proof : Let $V_1 = \{v \in V \mid \deg(v) \text{ is even}\}$,
$V_2 = \{v \in V \mid \deg(v) \text{ is odd}\}$.

$$2e = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

$$\Rightarrow \sum_{v \in V_2} \deg(v) \text{ is even.}$$

Directed Graph

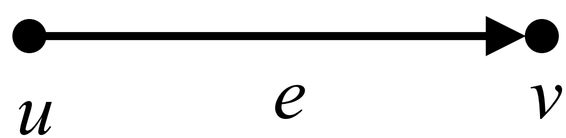
Def 3. $G = (V, E)$: directed graph,

$e = (u, v) \in E$: u is adjacent to v

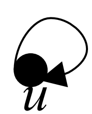
v is adjacent from u

- u : initial vertex of e

  v : terminal (end) vertex of e



The initial vertex and terminal vertex of a loop are the same



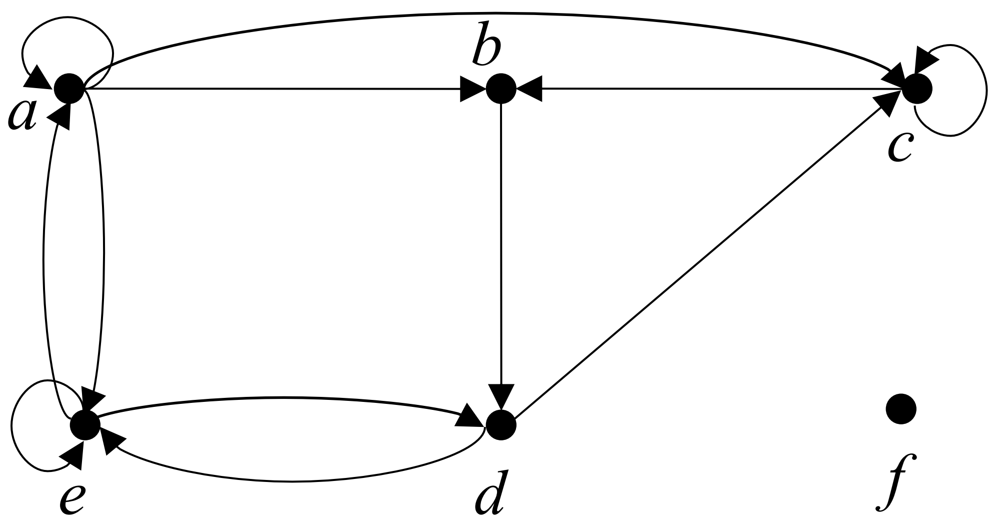
Directed Graph

$G = (V, E)$: directed graph, $v \in V$

$\deg^{-}(v)$: # of edges with v as a terminal.(in-degree)

$\deg^{+}(v)$: # of edges with v as a initial vertex. (out-degree)

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|$$



$\deg^{-}(a)=2$, $\deg^{+}(a)=4$

$\deg^{-}(b)=2$, $\deg^{+}(b)=1$

$\deg^{-}(c)=3$, $\deg^{+}(c)=2$

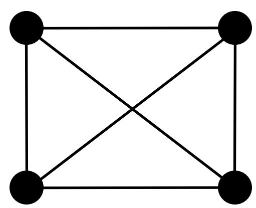
$\deg^{-}(d)=2$, $\deg^{+}(d)=2$

$\deg^{-}(e)=3$, $\deg^{+}(e)=3$

$\deg^{-}(f)=0$, $\deg^{+}(f)=0$

Regular Graph

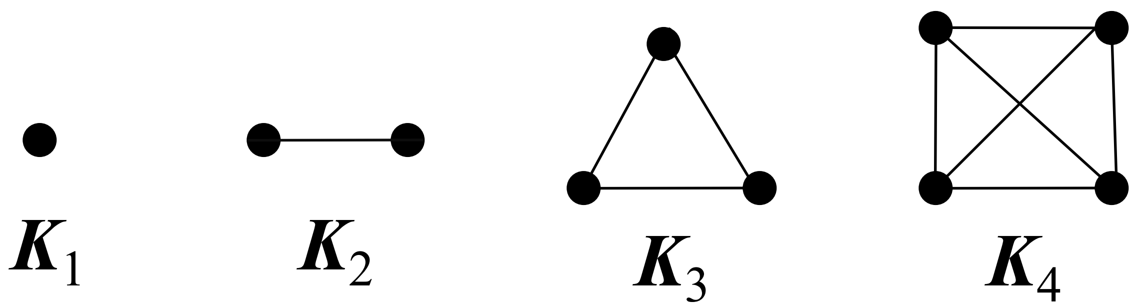
- A simple graph $G=(V, E)$ is called regular if every vertex of this graph has the same degree. A regular graph is called n-regular if $\deg(v)=n$, $\forall v \in V$.



is 3-regular.

Some Special Simple Graphs

- The complete graph on n vertices, denoted by $\boldsymbol{K_n}$, is the simple graph that contains exactly one edge between each pair of distinct vertices.

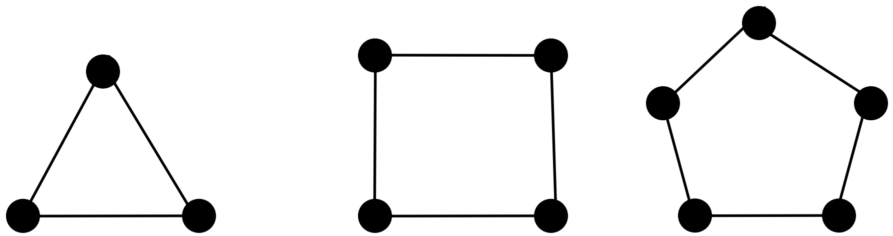


Note. K_n is $(n-1)$-regular, $|V(K_n)|=n$,

$$|E(K_n)| = \binom{n}{2}$$

Some Special Simple Graphs

- The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1,v_2\}, \{v_2,v_3\}, \ldots, \{v_{n-1},v_n\}, \{v_n,v_1\}$.



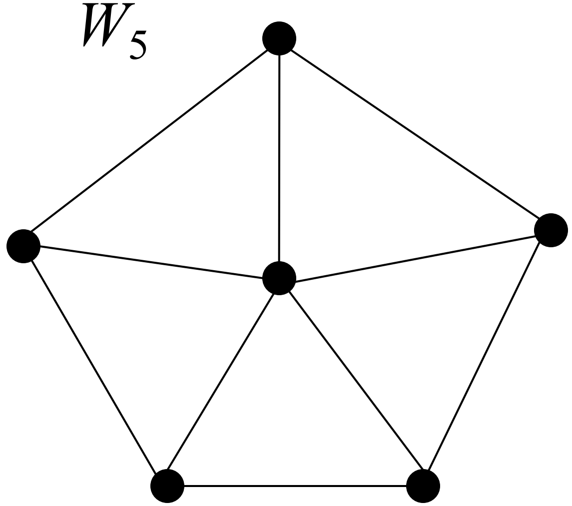
C_3 C_4 C_5

Note C_n is 2-regular, $|V(C_n)| = n$, $|E(C_n)| = n$

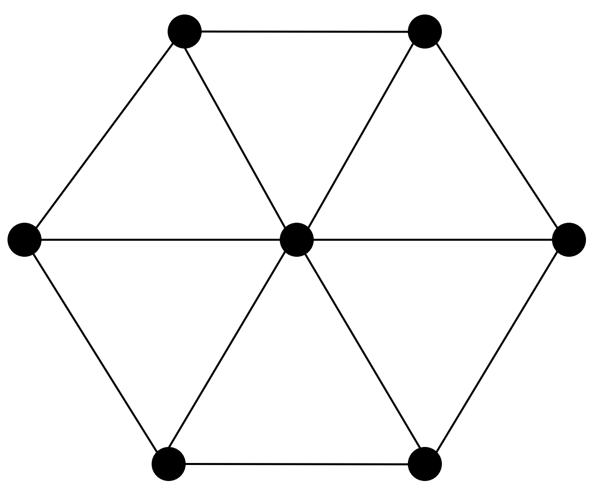
Some Special Simple Graphs

- We obtained the **wheel** W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

W_5



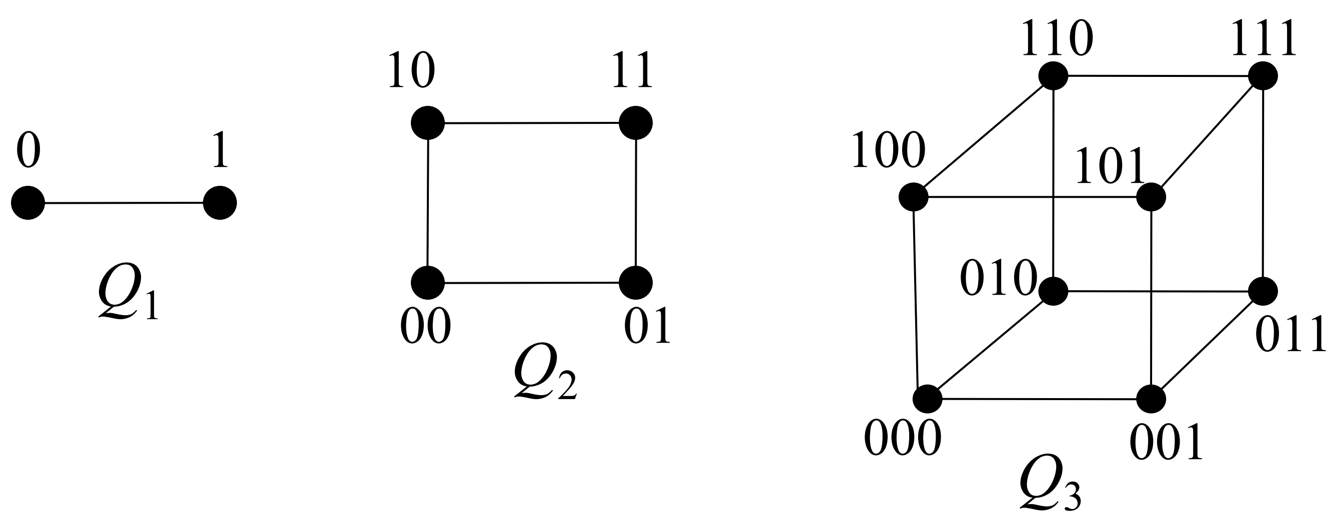
W_6



Note. $|V(W_n)| = n + 1$, $|E(W_n)| = 2n$,
W_n is not a regular graph if $n \neq 3$.

Some Special Simple Graphs

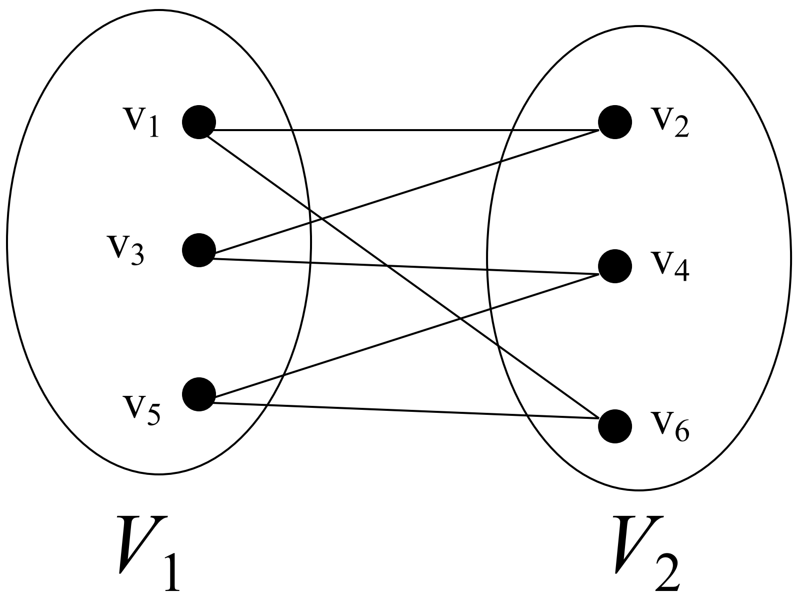
- The n-dimensional hypercube, or n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n.
  Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.



Note. Q_n is n-regular, $|V(Q_n)| = 2^n$, $|E(Q_n)| = (2^n n)/2 = 2^{n-1} n$

Some Special Simple Graphs: Bipartite

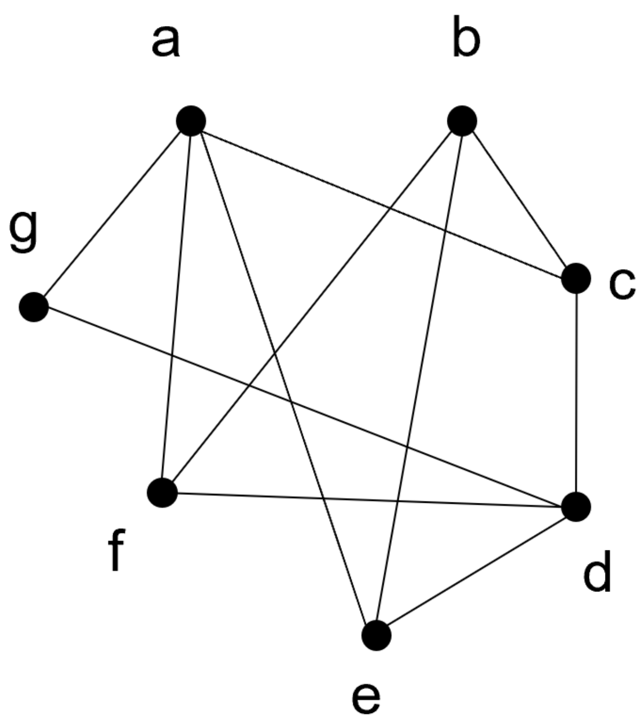
- A simple graph $G=(V,E)$ is called bipartite if V can be partitioned into V_1 and V_2, $V_1 \cap V_2 = \varnothing$, such that every edge in the graph connect a vertex in V_1 and a vertex in V_2.



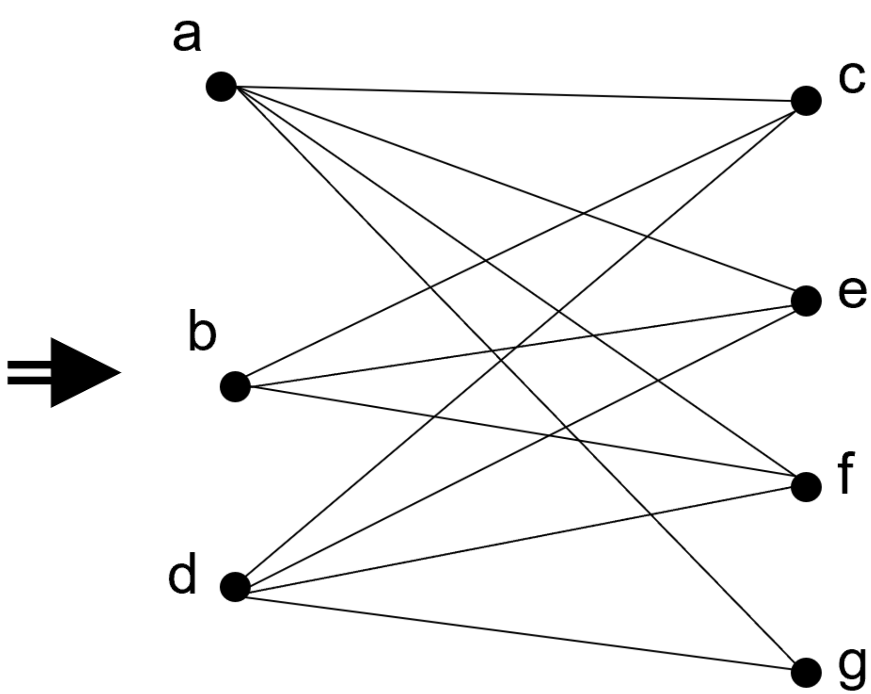
$\therefore$ C_6 is bipartite.

Some Special Simple Graphs: Bipartite

- Is the graph G bipartite ?



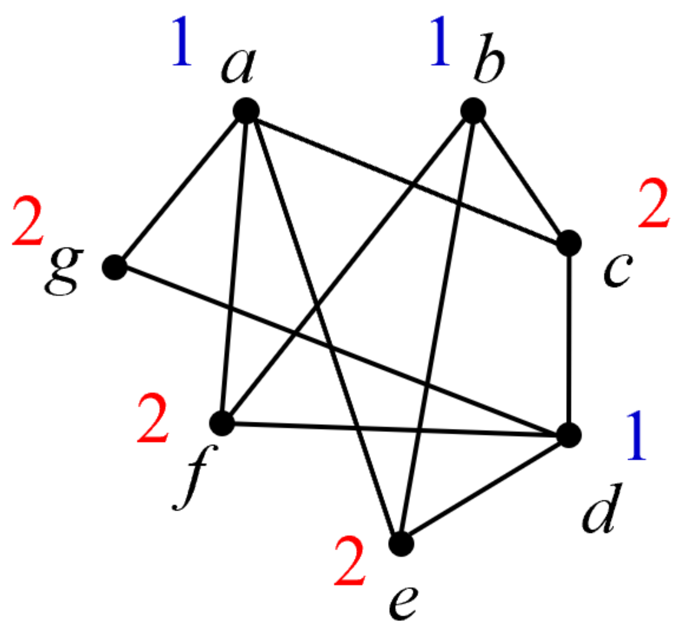
G



Yes !

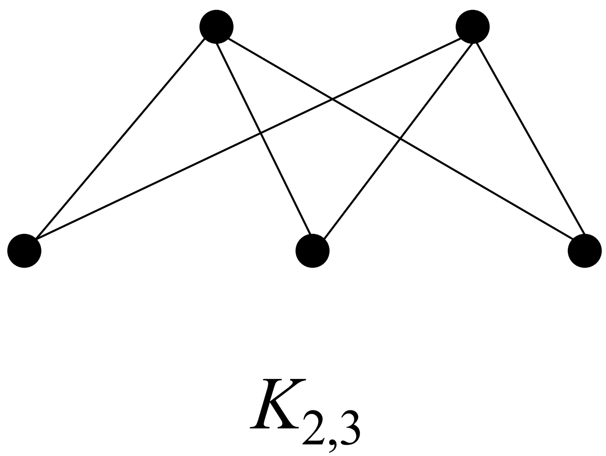
Some Special Simple Graphs: Bipartite

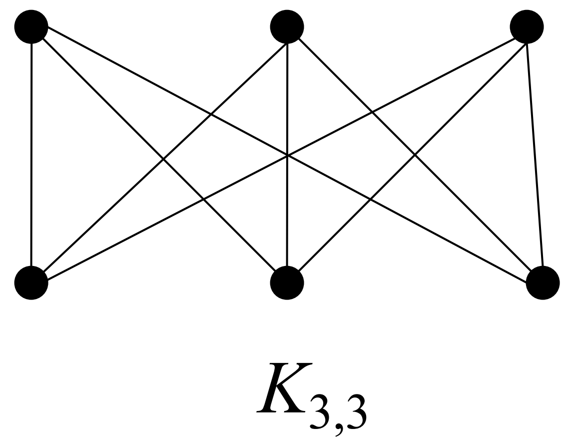
- A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.



Some Special Simple Graphs: Bipartite

- Complete Bipartite graphs ($K_{m,n}$)

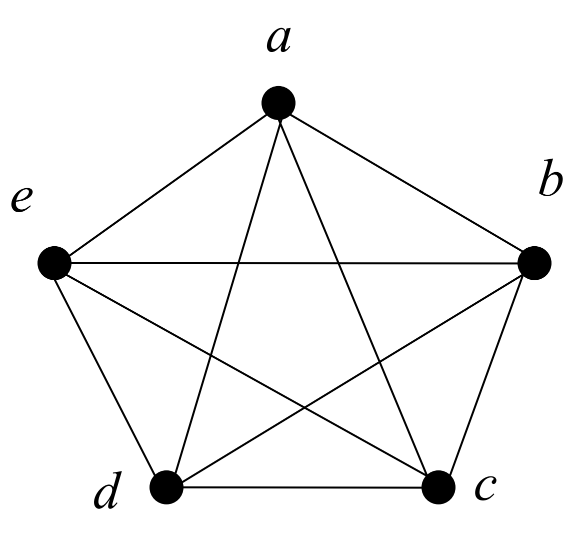




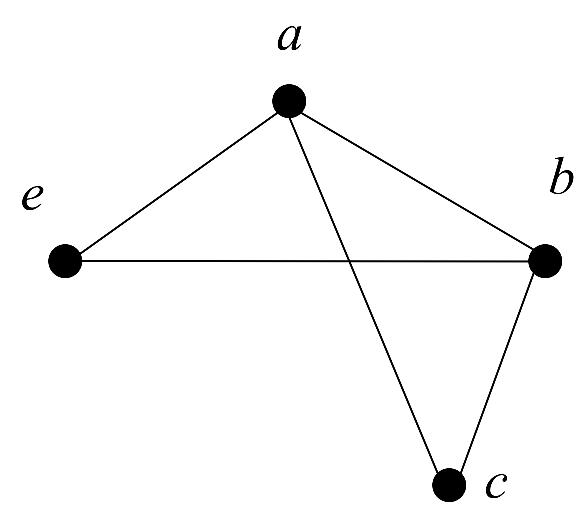
Note. $|V(K_{m,n})| = m+n$, $|E(K_{m,n})| = mn$,
$K_{m,n}$ is regular if and only if $m=n$.

New Graphs from Old

- A subgraph of a graph $G=(V, E)$ is a graph $H=(W, F)$ where $W \subseteq V$ and $F \subseteq E$. (Notice the f point w to connect)



K_5

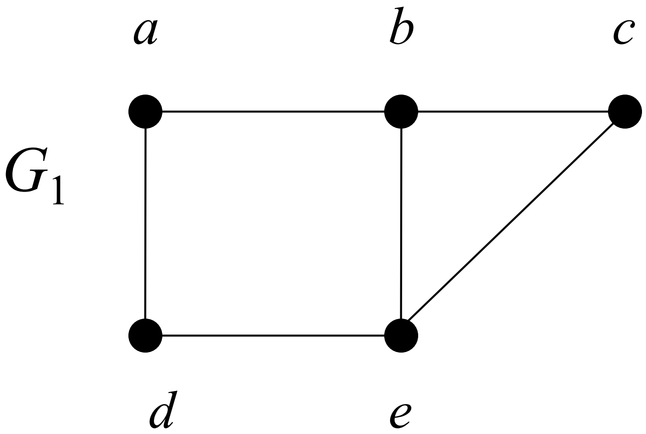


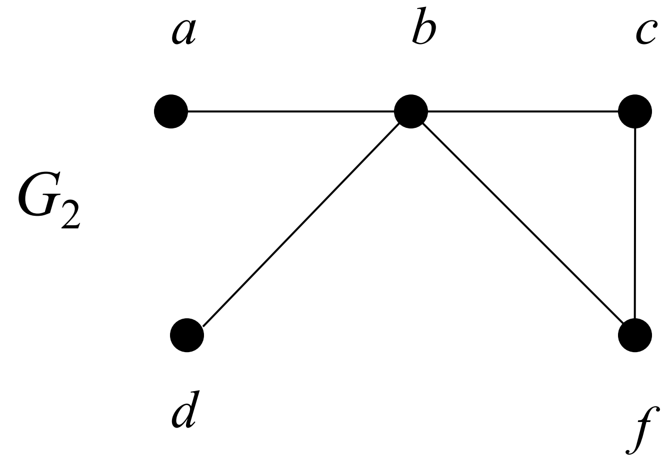
subgraph of K_5

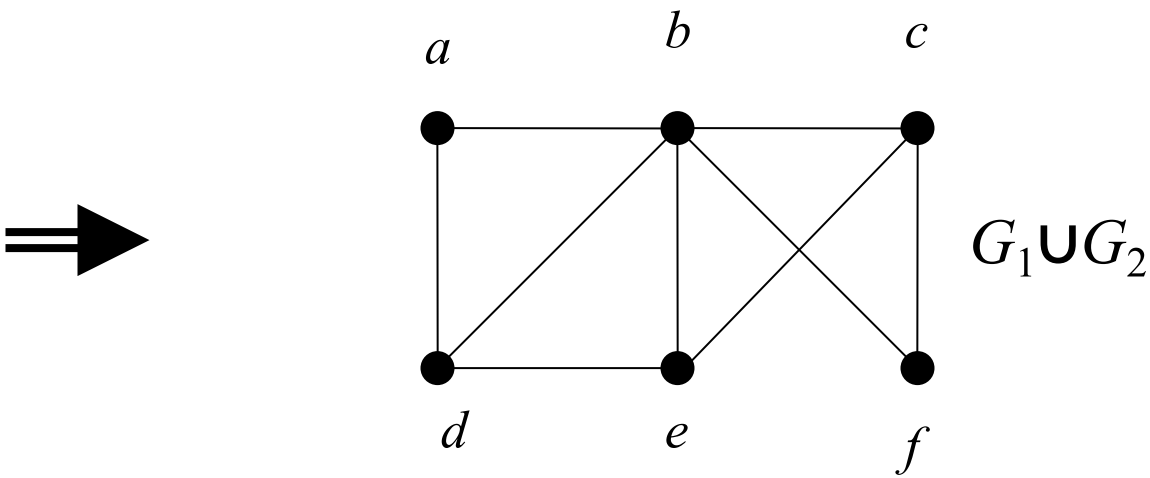
The union of two simple graphs

- $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is the simple graph

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$



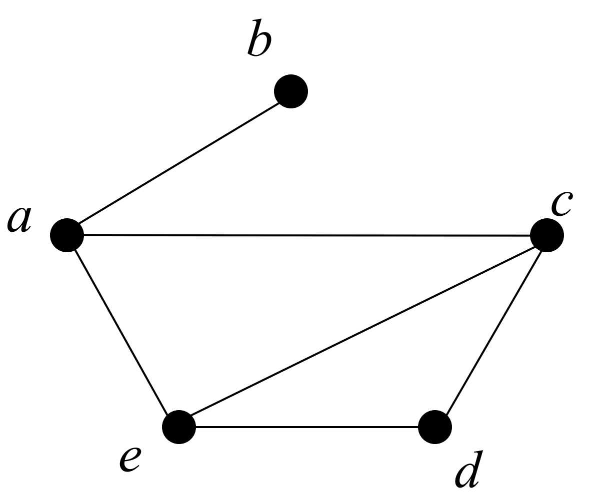




Representing Graphs and Graph Isomorphism

Adjacency list: Undirected graph

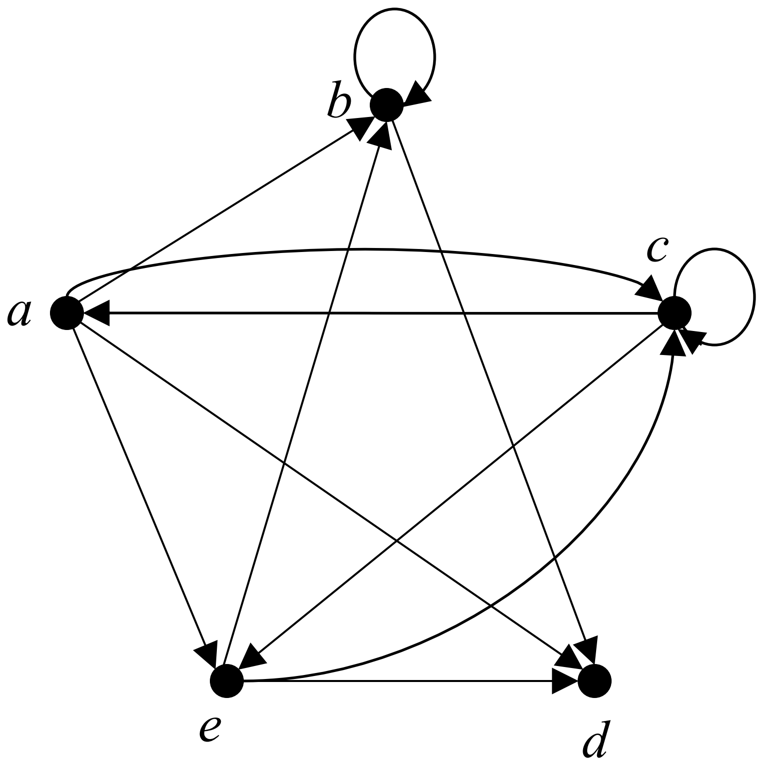
Example: Use adjacency lists to describe the simple graph given below.



Vertex	Adjacent Vertices
a	b,c,e
b	a
c	a,d,e
d	c,e
e	a,c,d

Representing Graphs and Graph Isomorphism

- **Adjacency list: Directed graph**



Initial vertex	Terminal vertices
a	b,c,d,e
b	b,d
c	a,c,e
d	
e	b,c,d

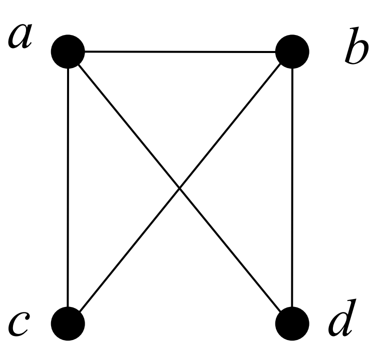
Representing Graphs and Graph Isomorphism

Adjacency Matrices

$G=(V, E)$: simple graph, $V=\{v_1,v_2,\ldots,v_n\}$.

A matrix A is called the adjacency matrix of G

if $A=[a_{ij}]_{n\times n}$, where $a_{ij}=$ 1, if $\{v_i,v_j\}\in E$, and 0 otherwise.



$$A_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ b \\ d \\ c \\ a \end{array} \begin{array}{c} \begin{array}{cccc} b & d & c & a \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

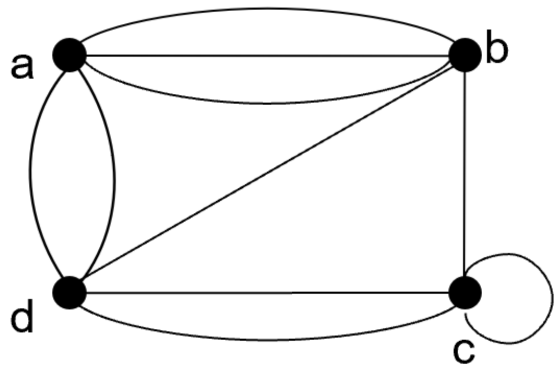
Note:

1. There are n! different adjacency matrices for a graph with n vertices.
2. The adjacency matrix of an undirected graph is symmetric.
3. $a_{ii}=0$ (simple matrix has no loop)

Representing Graphs and Graph Isomorphism

Adjacency Matrices

(Pseudograph) (Matrix may not be 0,1 matrix.)



$$\Longrightarrow \quad A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

If $A=[a_{ij}]$ is the adjacency matrix for the directed graph, then

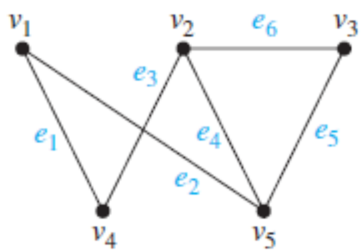
$$a_{ij} = \begin{cases} 1 & \text{, if } v_i \longrightarrow v_j \\ 0 & \text{, otherwise} \end{cases}$$

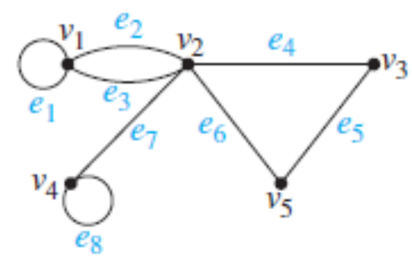
(So the matrix is not necessarily symmetrical)

Representing Graphs and Graph Isomorphism

- **Incidence Matrices**
- Let $G=(V, E)$: be an undirected graph. Suppose that $v_1,v_2,\ldots,v_n$ are the vertices and $e_1,e_2,\ldots,e_n$ are the edges of G . Then the incidence matrix with respect to this ordering of V and E is the n x m matrix $M=[m_{ij}]$, where

$$m_{i,j} = \begin{cases} 1 \text{ when edge } e_j \text{ is incident with } v_i, \\ 0 \text{ otherwise.} \end{cases}$$





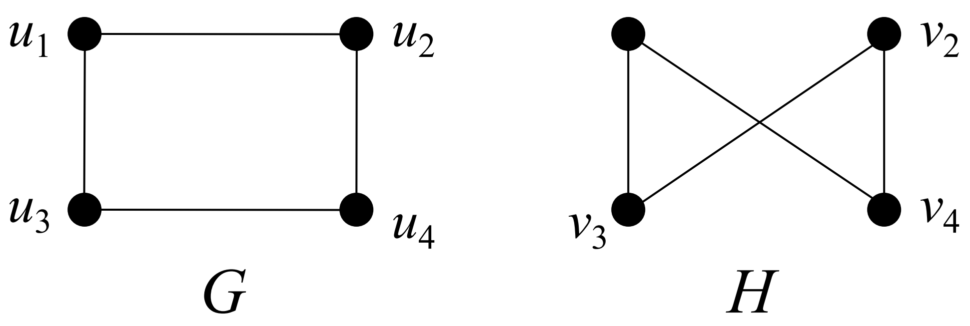
	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	1	0	0	0	0	0
v_2	0	1	1	1	0	1	1	0
v_3	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1
v_5	0	0	0	0	1	1	0	0

Representing Graphs and Graph Isomorphism

- **Isomorphism of Graphs**

The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is an one-to-one and onto function f from V_1 to V_2 with the property that $a \sim b$ in G_1 iff $f(a) \sim f(b)$ in G_2, $\forall a,b \in V_1 f$, f is called an isomorphism.

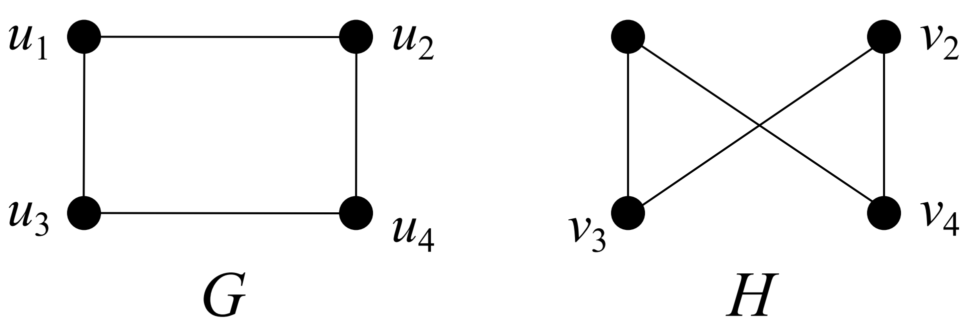


G is isomorphic to H

Representing Graphs and Graph Isomorphism

- Show that G and H are isomorphic.

Solution: The function f with $f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3,$ and $f(u_4) = v_2$ is a one-to-one correspondence between $V(G)$ and $V(H)$.



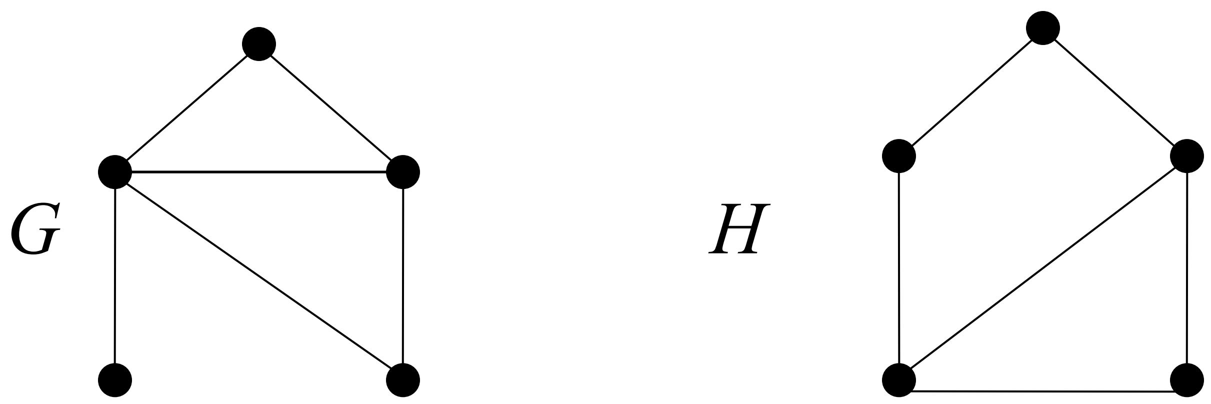
G is isomorphic to H

Isomorphism graphs there will be:

1. The same number of points (vertices)
2. The same number of edges
3. The same number of degree

Representing Graphs and Graph Isomorphism

- Show that G and H are not isomorphic.

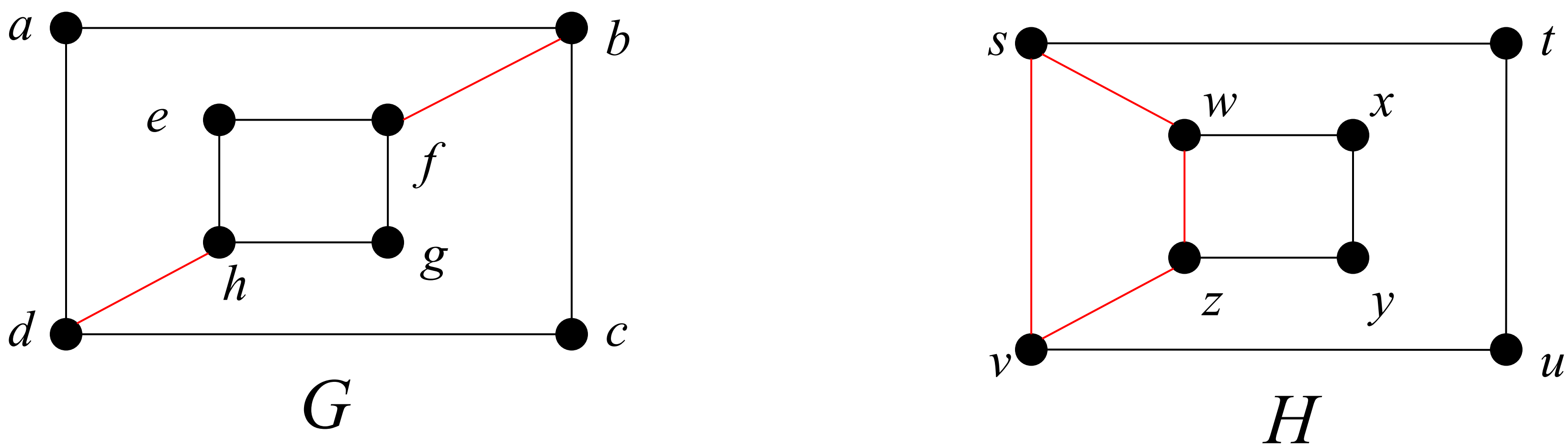


Solution :

G has a vertex of degree $= 1$, H don't

Representing Graphs and Graph Isomorphism

- Determine whether G and H are isomorphic.



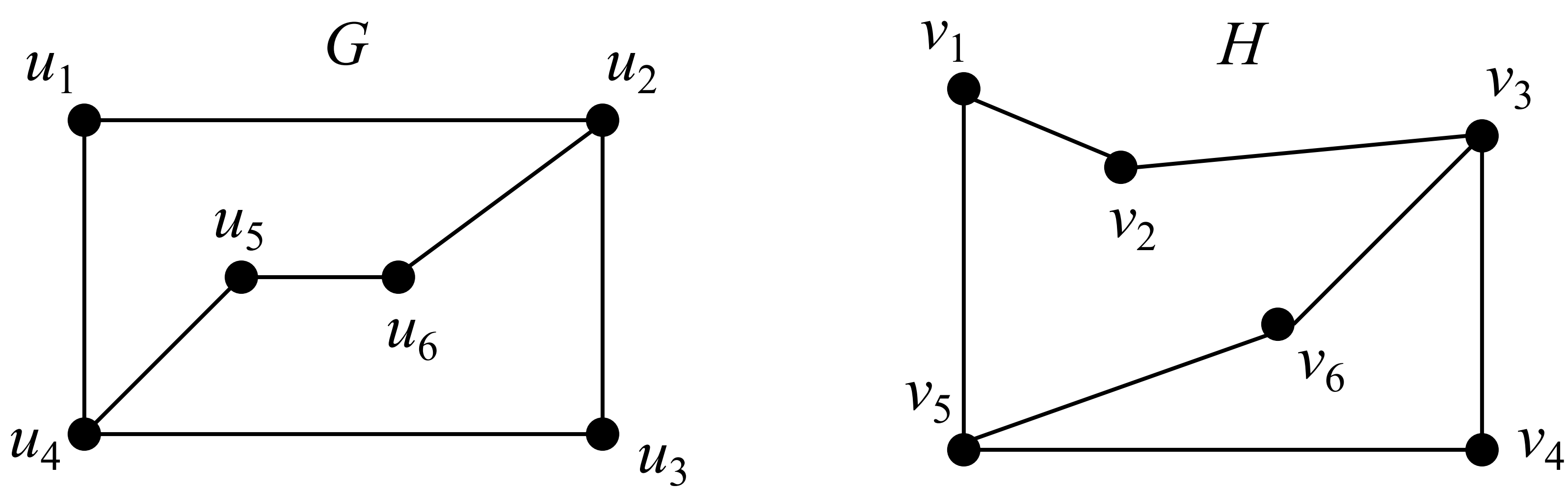
Solution : ∵ In G, deg(a)=2, which must correspond to either t, u, x, or y in H degree

Each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G

∴ G and H are not isomorphic.

Representing Graphs and Graph Isomorphism

- Determine whether the graphs G and H are isomorphic.



Solution:

$f(u_1)=v_6, f(u_2)=v_3, f(u_3)=v_4, f(u_4)=v_5, f(u_5)=v_1, f(u_6)=v_2$

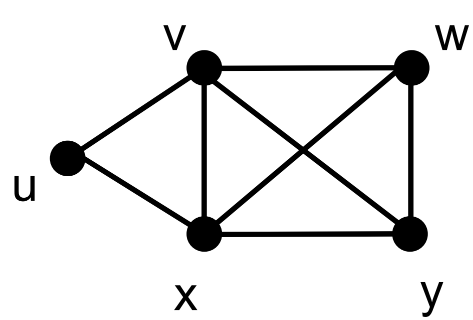
$\Rightarrow$ Yes

Euler and Hamiltonian Paths and Shortest-Path Problems

Connectivity

- In an undirected graph, a path of length n from u to v is a sequence of $n+1$ adjacent vertices going from vertex u to vertex v.
  (e.g., P: $u=x_0, x_1, x_2, \ldots, x_n=v$.) (P has n edges.)
- path: Points and edges in unrepeatable
  trail: Allows duplicate path (not repeatable)
  walk: Allows point and duplicate path

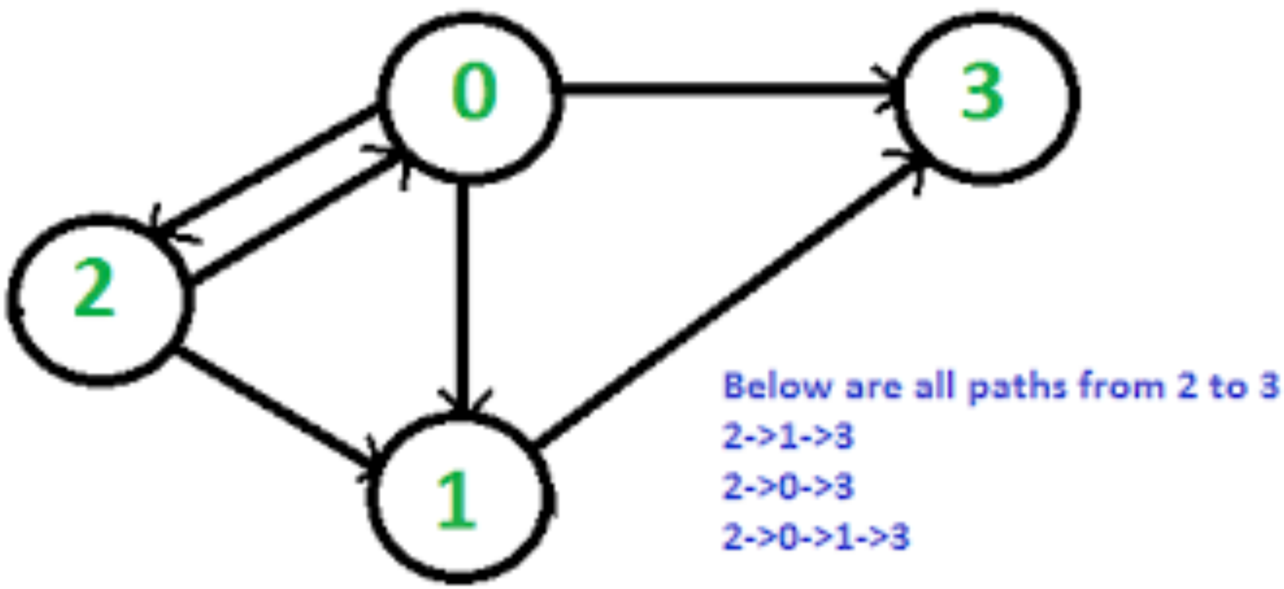
  cycle: path with $u=v$



path: u, v, y
trail: u, v, w, y, v, x, y
walk: u, v, w, v, x, v, y
cycle: u, v, y, x, u

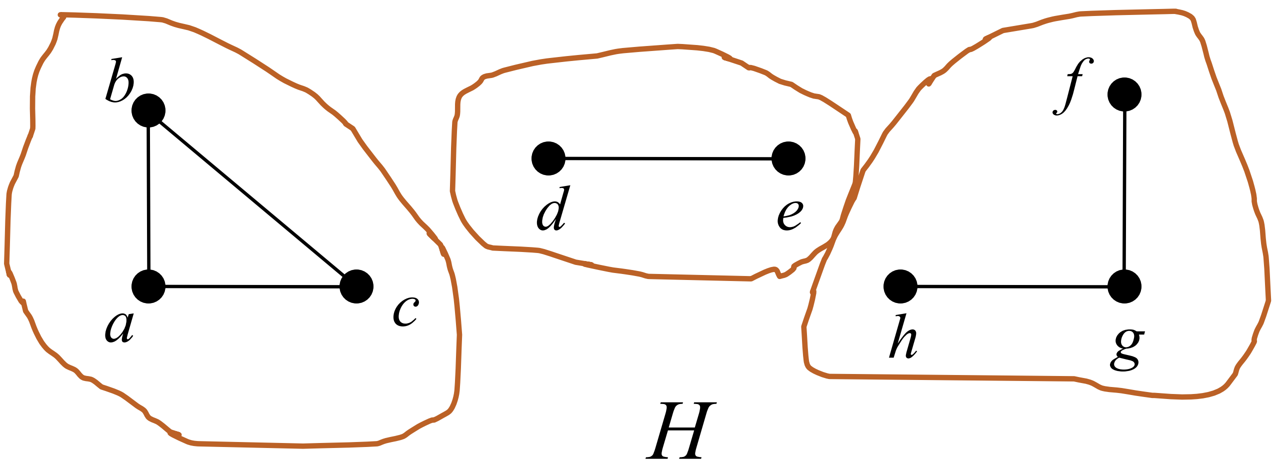
Paths in Directed Graphs

- The same as in undirected graphs, but the path must go in the direction of the arrows.



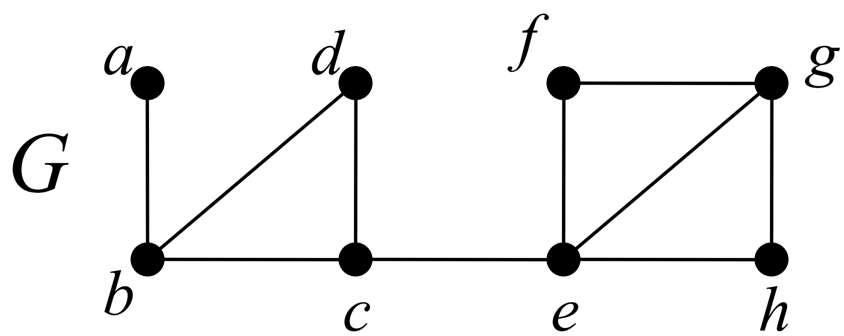
Connectedness in Undirected Graphs

- An undirected graph is *connected* if there is a path between every pair of distinct vertices in the graph.
- *Connected component*: maximal connected subgraph. (An unconnected graph will have several component)



Cut Vertex and Cut Edge

- A *cut vertex* separates one connected component into several components if it is removed.
  A *cut edge* separates one connected component into two components if it is removed.



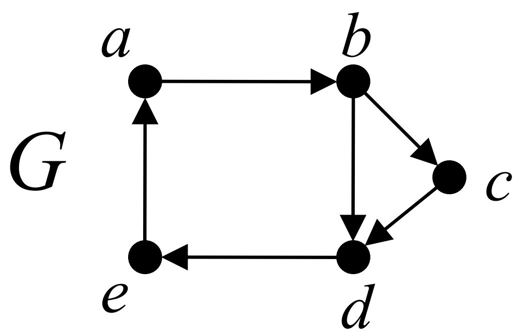
cut vertices: b, c, e

cut edges:

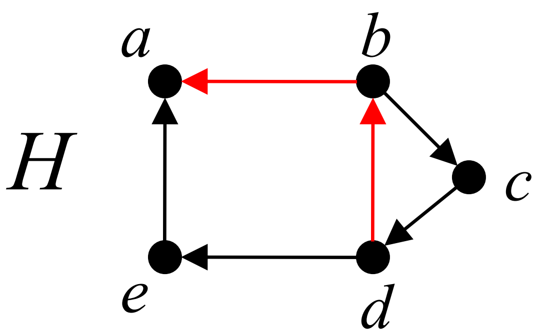
$\{a, b\}, \{c, e\}$

Connectedness in Directed Graphs

- A directed graph is *strongly connected* if there is a path from a to b for any two vertices a, b.
  A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graphs.



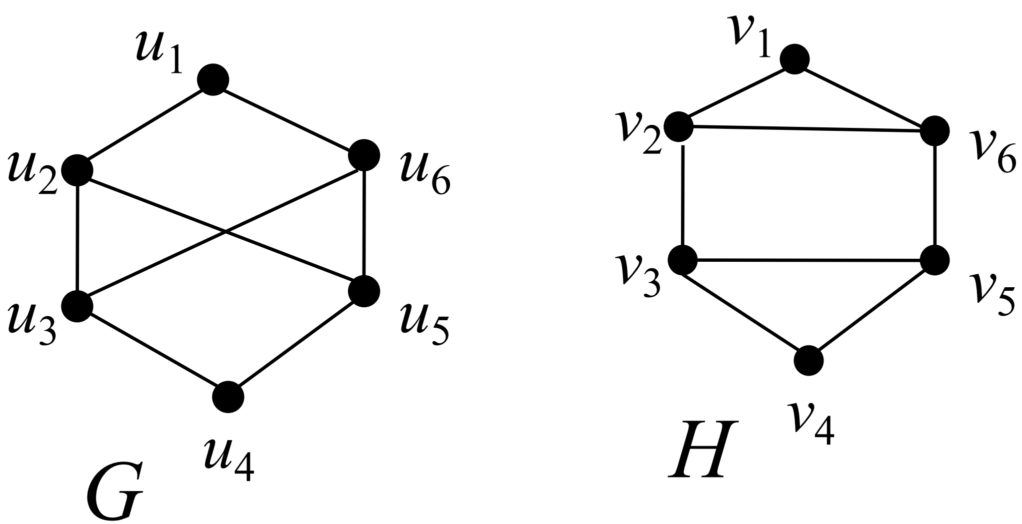
strongly connected



weakly connected

Paths and Isomorphism

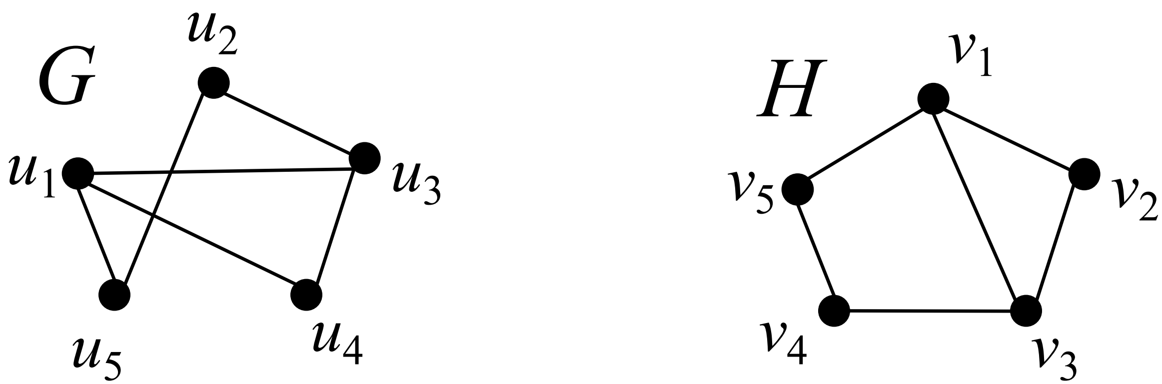
- Note that connectedness, and the existence of a circuit or simple circuit of length *k* are graph invariants with respect to isomorphism.
- Determine whether the graphs G and H are isomorphic.



Solution: No, Because G has no simple circuit of length three, but H does.

Paths and Isomorphism

- Determine whether the graphs G and H are isomorphic.



Solution:

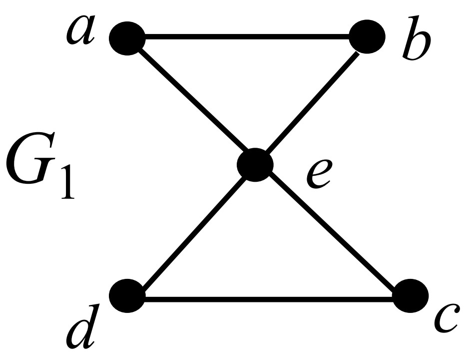
Both G and H have 5 vertices, 6 edges, two vertices of deg 3, three vertices of deg 2, a 3-cycle, a 4-cycle, and a 5-cycle.
$\Rightarrow$ G and H may be isomorphic.
The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$ and $f(u_5) = v_5$ is a one-to-one correspondence between $V(G)$ and $V(H)$. $\Rightarrow$ G and H are isomorphic.

Euler and Hamilton Paths

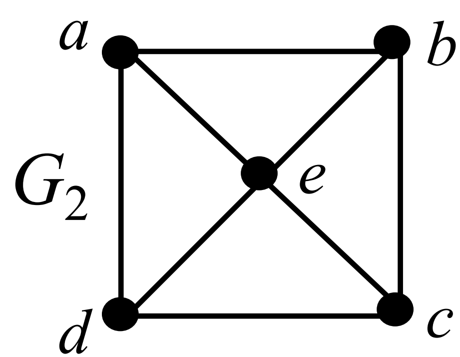
- An *Euler circuit* in a graph G is a simple circuit containing every edge of G.
  An *Euler path* in G is a simple path containing every edge of G.
- A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.
- A connected multigraph has an Euler path (but not an Euler circuit) if and only if it has exactly 2 vertices of odd degree.

Example

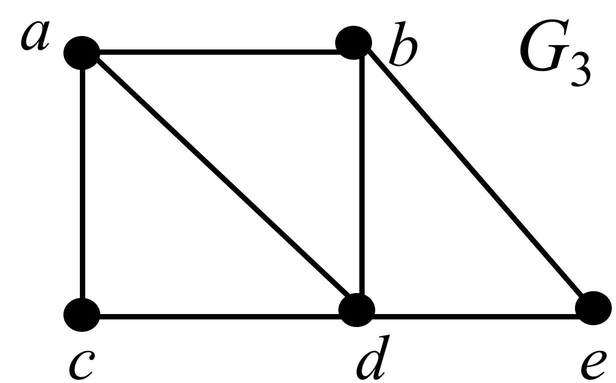
- Which of the following graphs have an Euler circuit or an Euler path?



Euler circuit



none

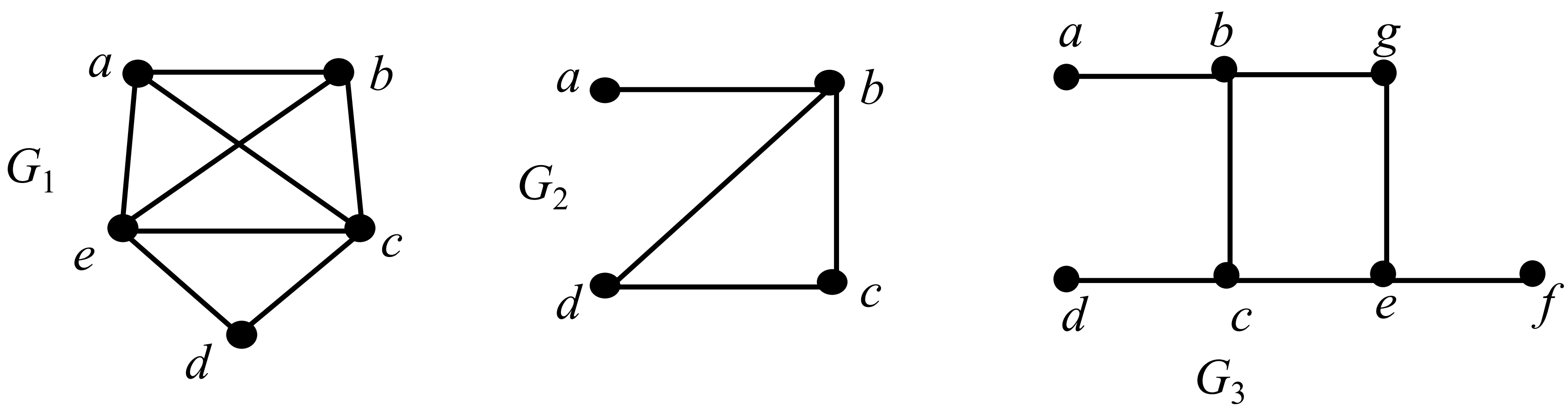


Euler path

Hamilton Paths and Circuits

- A *Hamilton path* is a path that traverses each vertex in a graph G exactly once.
  A *Hamilton circuit* is a circuit that traverses each vertex in G exactly once.

- Which of the following graphs have a Hamilton circuit or a Hamilton path?



Hamilton circuit: G_1 Hamilton path: G_1, G_2

Shortest-Path Problems

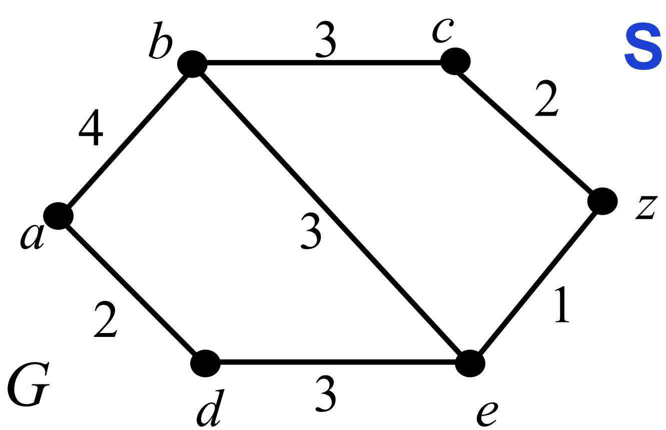
- Graphs that have a number assigned to each edge are called *weighted graphs*.
- The length of a path in a weighted graph is the sum of the weights of the edges of this path.

Shortest path Problem:

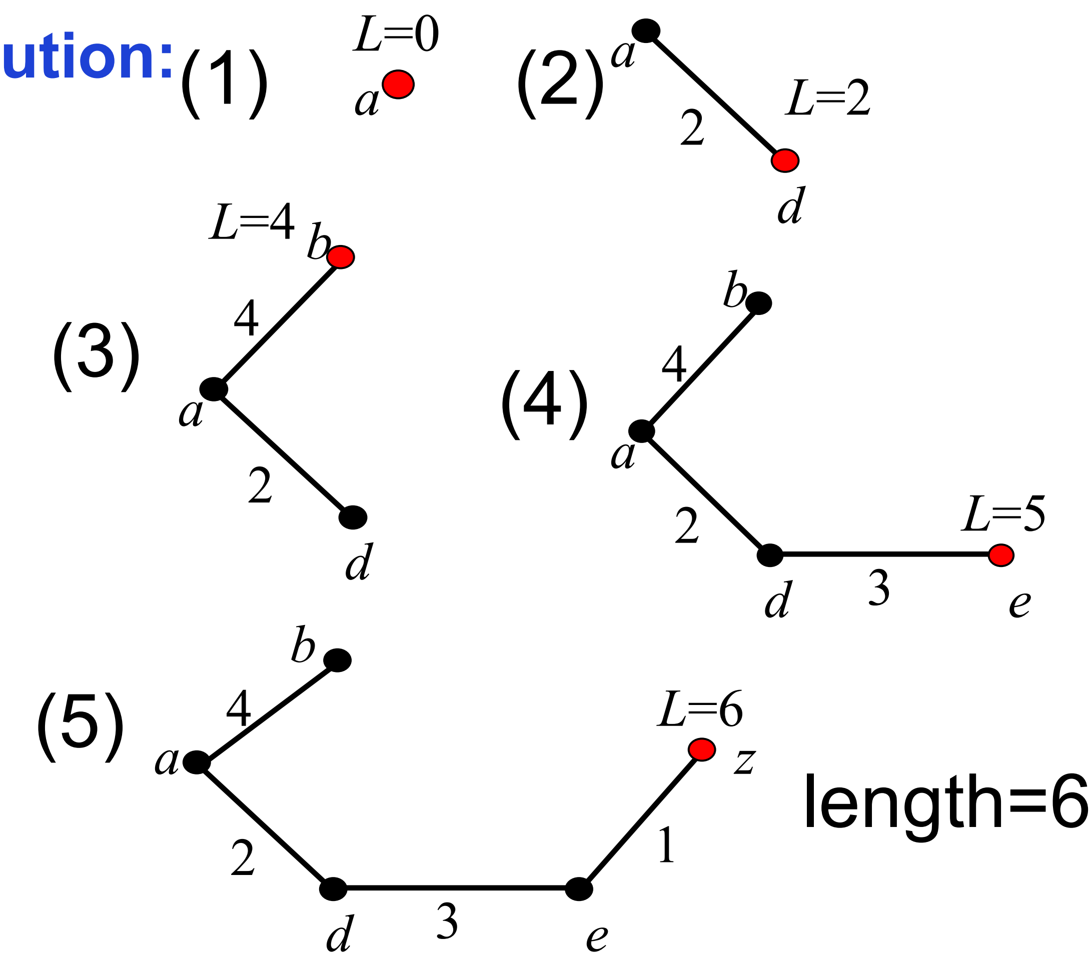
Determining the path of least sum of the weights between two vertices in a weighted graph.

Exampe

- What is the length of a shortest path between a and z in the weighted graph G?



Solution:



Dijkstra's Algorithm

Procedure *Dijkstra*(G: weighted connected simple graph, with all weights positive)

{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and weights $w(v_i, v_j)$ where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}

for $i := 1$ **to** n
    $L(v_i) := \infty$

$L(a) := 0$

$S := \varnothing$

while $z \notin S$

begin

    $u :=$ a vertex not in S with $L(u)$ minimal

    $S := S \cup \{u\}$

    for all vertices v not in S

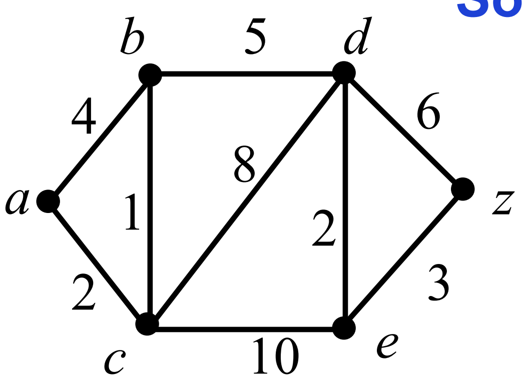
        if $L(\mathrm{u}) + w(u, v) < L(v)$ **then** $L(v) := L(\mathrm{u}) + w(u, v)$

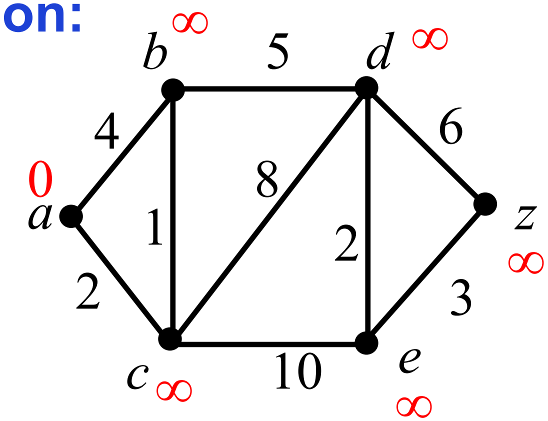
end $\{L(z) =$ length of a shortest path from a to $z\}$

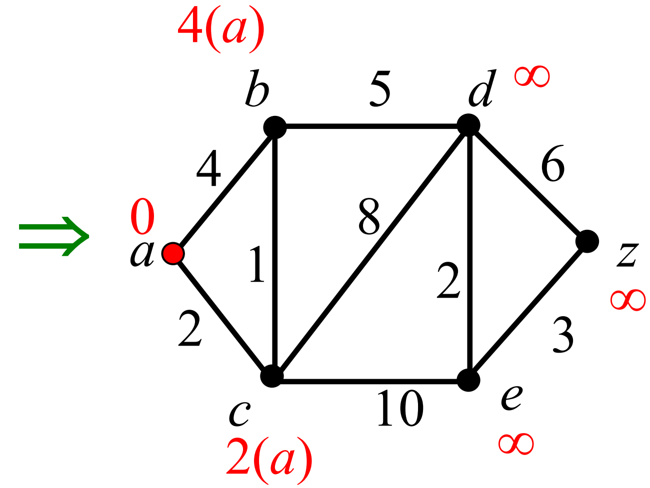
Example

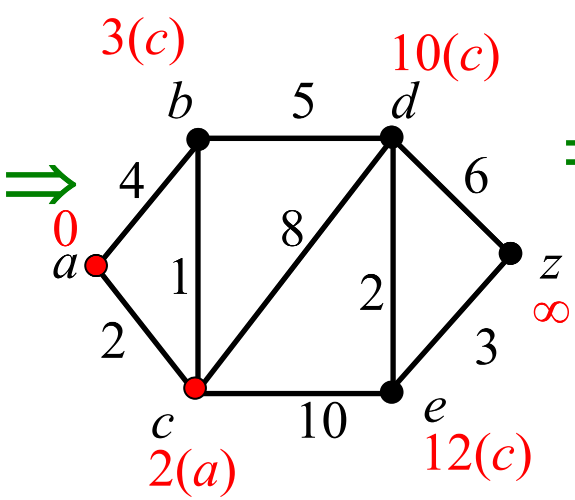
- Use Dijkstra's algorithm a shortest path between a and z.

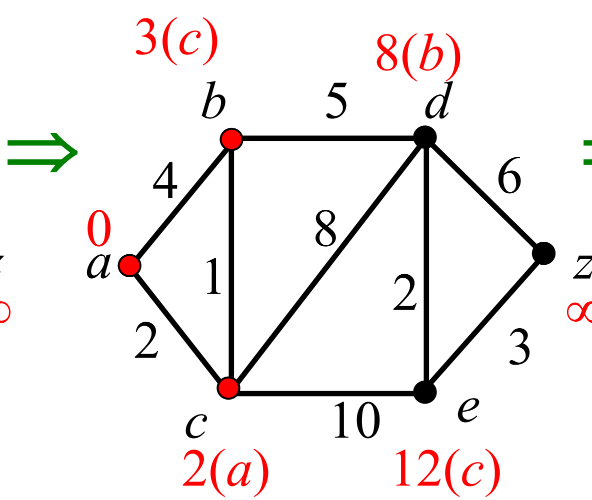
Solution:

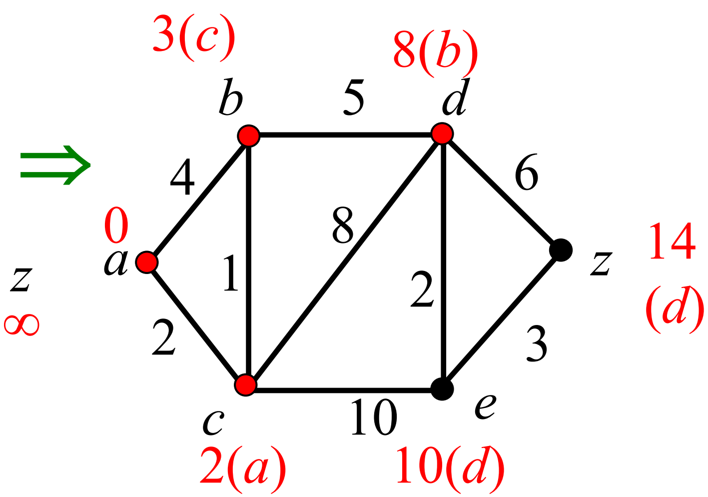




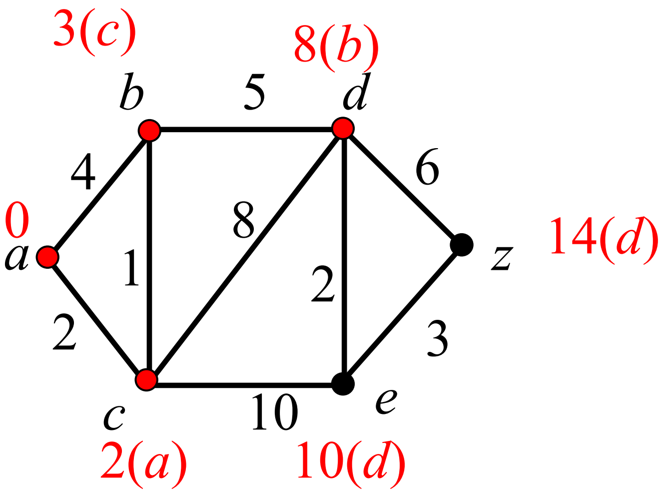




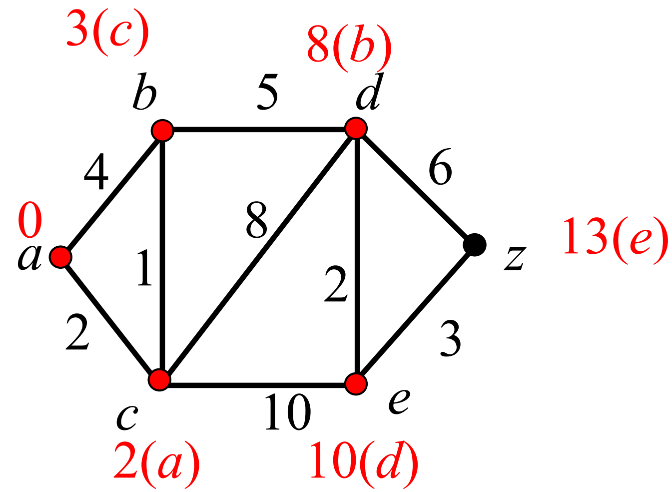




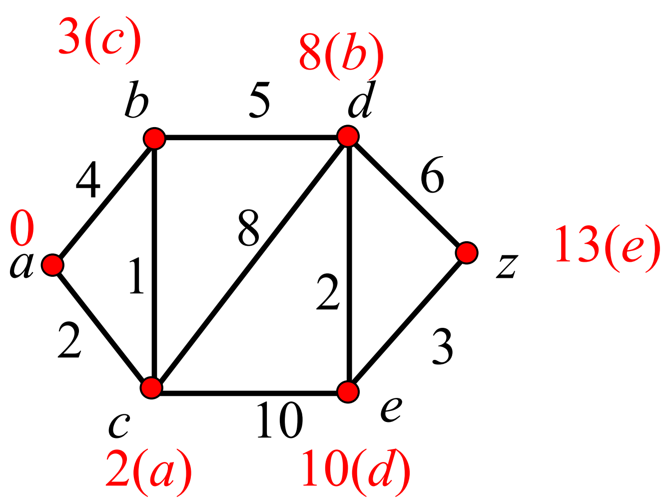
Example: Cont.



⇒



⇒



⇒ path: a, c, b, d, e, z
length: 13

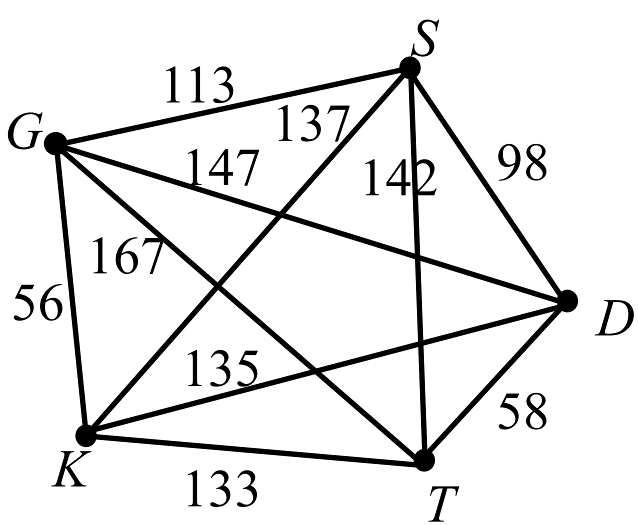
Remarks on Dijkstra's Algorithm

- Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.
- Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with n vertices.

The Traveling Salesman Problem

- A traveling salesman wants to visit each of n cities exactly once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?

Example (starting point D)



$D \to T \to K \to G \to S \to D$: 458

$D \to T \to S \to G \to K \to D$: 504

$D \to T \to S \to K \to G \to D$: 540

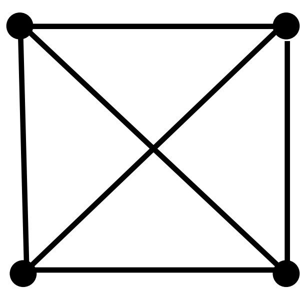
…

Planar Graphs and Graph Coloring

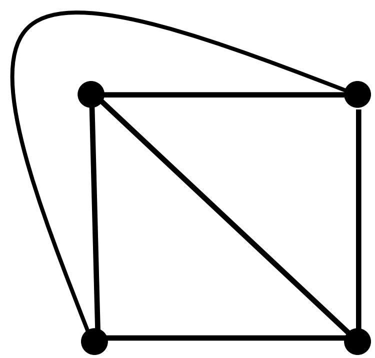
Planar Graphs

- A graph is called *planar* if it can be drawn in the plane without any edge crossing. Such a drawing is called a *planar representation* of the graph.

- Is K_4 planar?



K_4

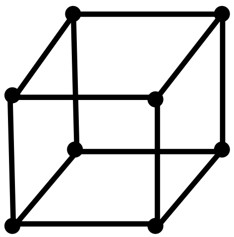


K_4 drawn with no crossings

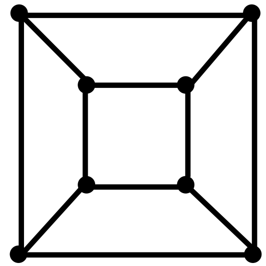
$\therefore$ K_4 is planar

Example

- Is Q_3 planar?



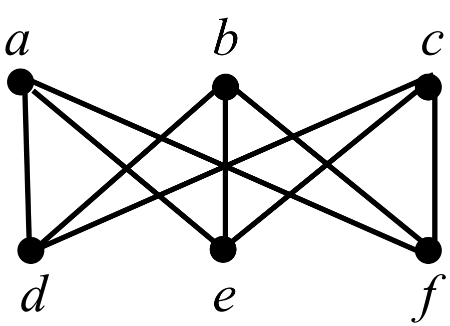
Q_3



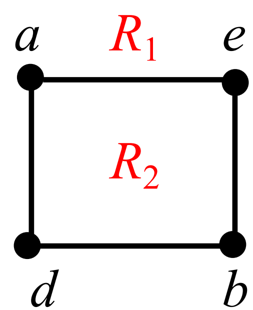
Q_3 drawn with no crossings

$\therefore$ Q_3 is planar

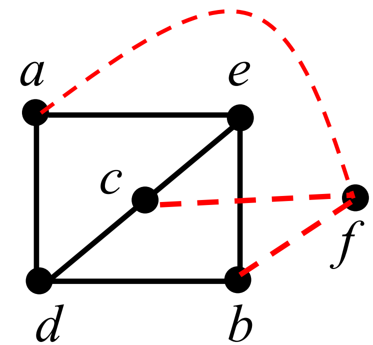
- Show that $K_{3,3}$ is nonplanar.



Solution:



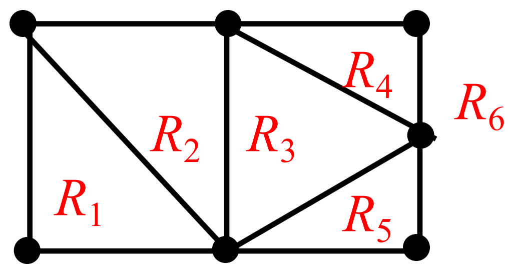
In any drawing, aebd is cycle, and will cut the plane into two regions



Regardless of which region c, could no longer put the f in that side staggered

Euler's Formula

- A planar representation of a graph splits the plane into regions, including an unbounded region.
- **Example :** How many regions are there in the following graph?



Solution: 6

Euler's Formula

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Example

- Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?
- **Solution:**

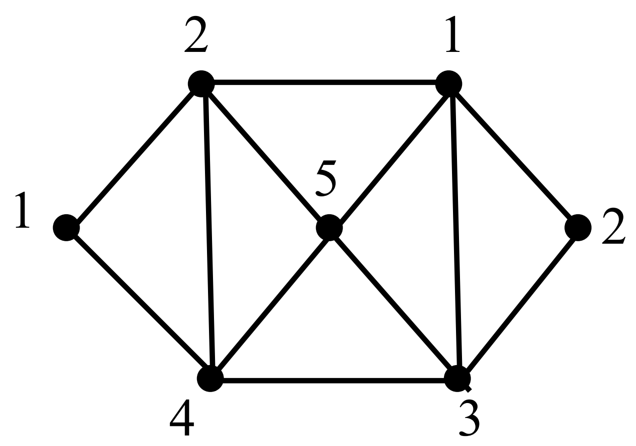
$$v = 20,\ 2e = 3\times 20 = 60,\ e = 30$$

$$r = e - v + 2 = 30 - 20 + 2 = 12$$

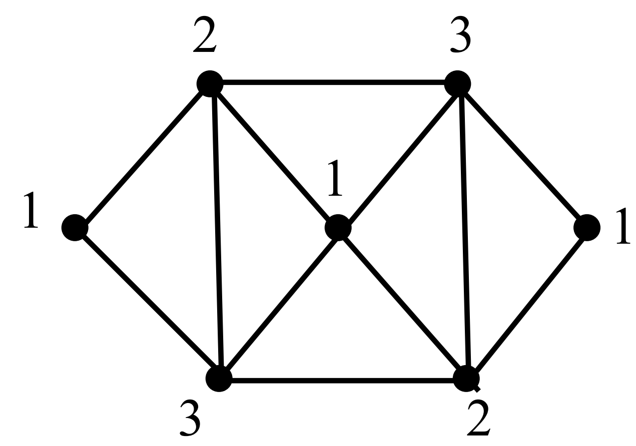
Graph Coloring

- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Example:



5-coloring



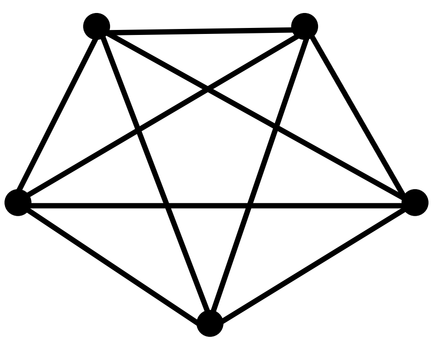
3-coloring

Less the number of colors, the better

Graph Coloring

- The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph. (denoted by $\chi(G)$)

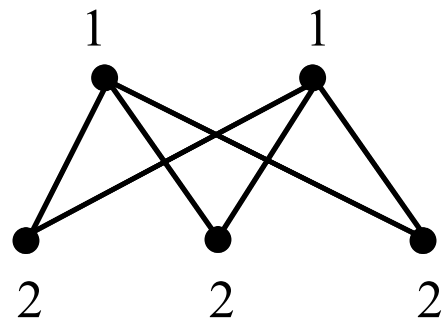
Example: $\chi(K_5)=5$



Note: $\chi(K_n)=n$

Example

$\chi(K_{2,3}) = 2.$

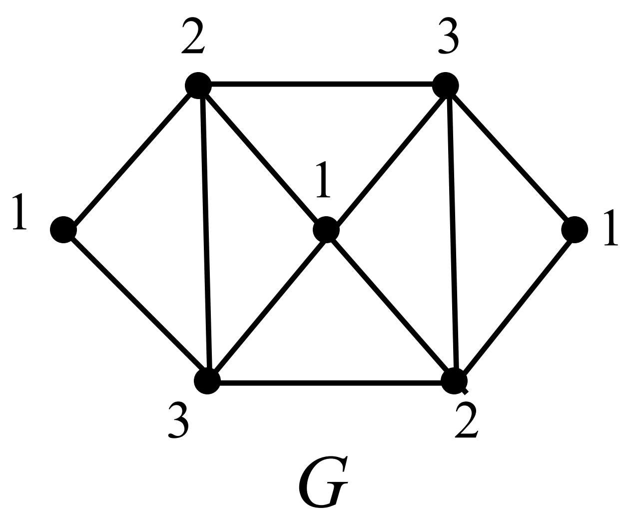


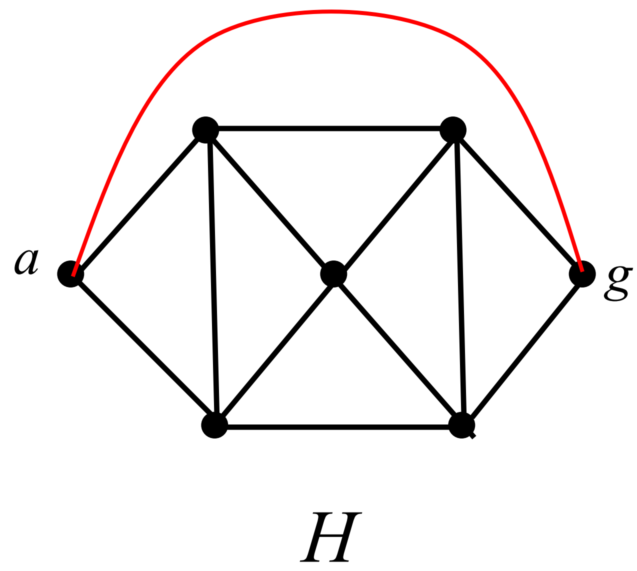
Note: $\chi(K_{m,n}) = 2$

Note: If G is a bipartite graph, $\chi(G) = 2.$

Example

- What are the chromatic numbers of the graphs G and H?





Solution: G has a 3-cycle
$\Rightarrow \chi(G) \geq 3$

G has a 3-coloring
$\Rightarrow \chi(G) \leq 3$

$\Rightarrow \chi(G) = 3$

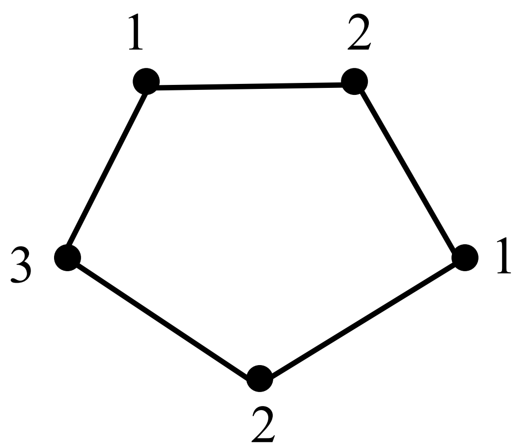
Solution: any 3-coloring for $H - \{(a,g)\}$ gives the same color to a and g
$\Rightarrow \chi(H) > 3$

4-coloring exists $\Rightarrow \chi(H) = 4$

Example

$\chi(C_n) = 2$ if n is even, $\chi(C_n) = 3$ if n is odd.

C_n is bipartite
when n is even.



Theorem 1. (The Four Color Theorem)
The chromatic number of a planar graph is no greater than four.

Corollary
Any graph with chromatic number >4 is nonplanar.

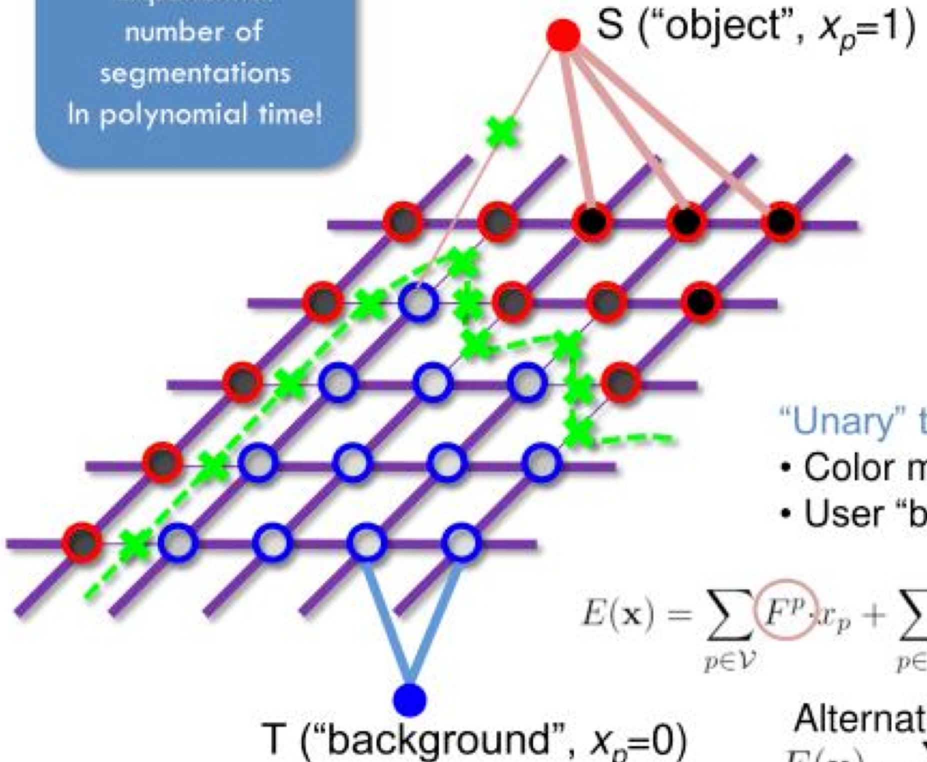
Applications of Graph

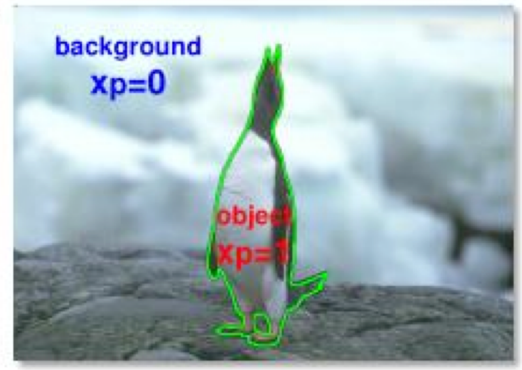
Image segmentation

Graph cut segmentation

[Boykov and Jolly, 2001]

Exponential number of segmentations In polynomial time!





"Unary" terms:
- Color models
- User "brushes"

"Pairwise" terms:
- Ising prior
- Edge cues

$$E(\mathbf{x}) = \sum_{p\in\mathcal{V}} F^p \cdot x_p + \sum_{p\in\mathcal{V}} B^p \cdot (1-x_p) + \sum_{p,q\in\mathcal{E}} P^{pq} \cdot |x_p - x_q|$$

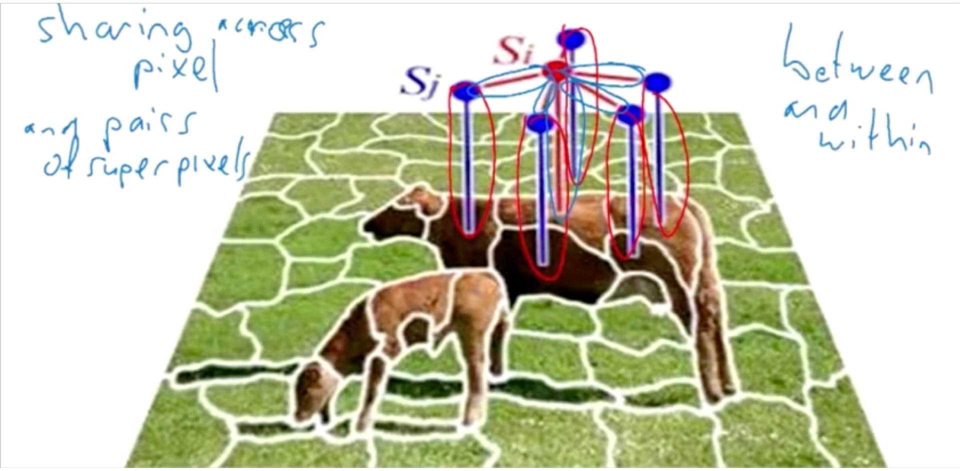
Alternative notation:

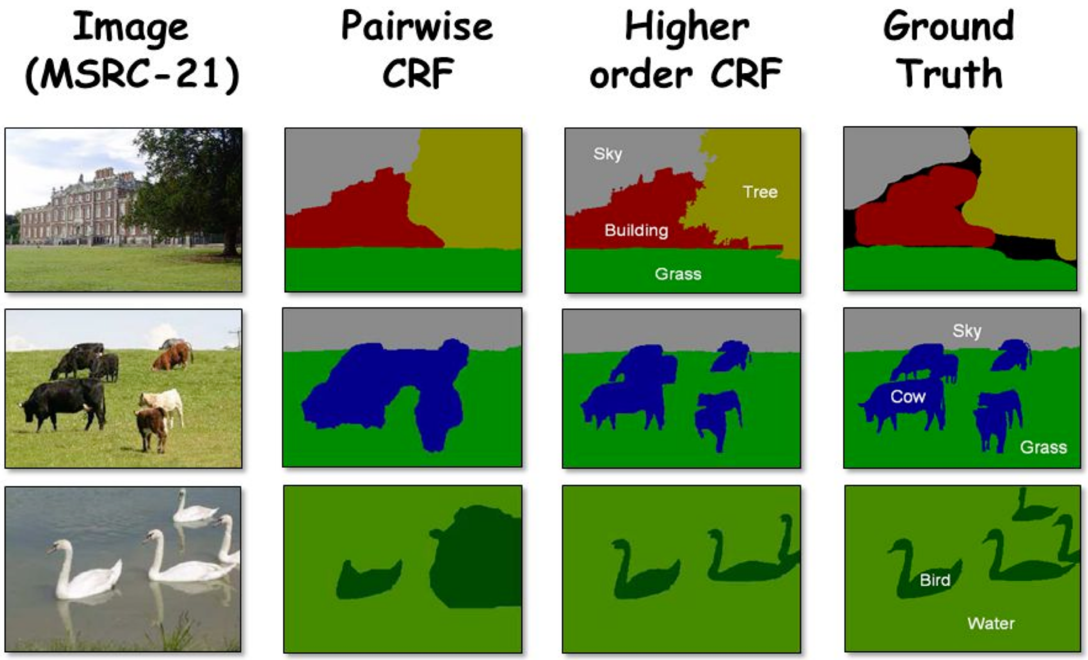
$$E(\mathbf{x}) = \sum_{p\in\mathcal{B}} U^p \cdot x_p + \sum_{\{p,q\}\in\mathcal{E}} V^{pq} \cdot |x_p - x_q|$$

Microsoft Research Cambridge

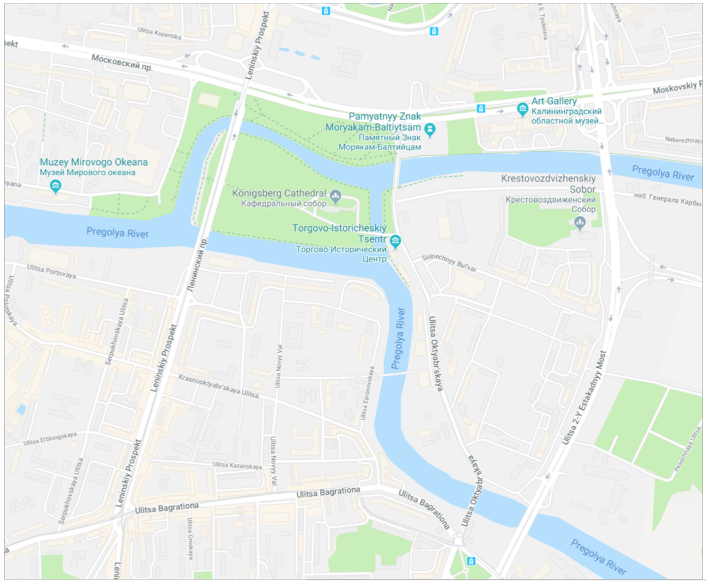
Image Segmentation beyond Graph Cuts

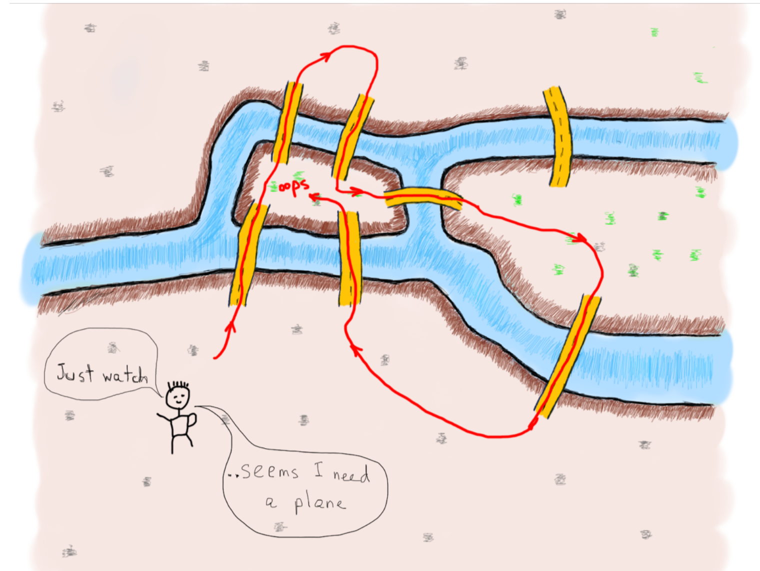
CRF Model for Image Segmentation

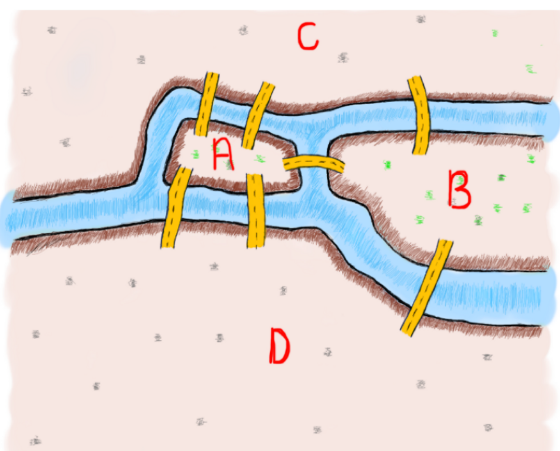


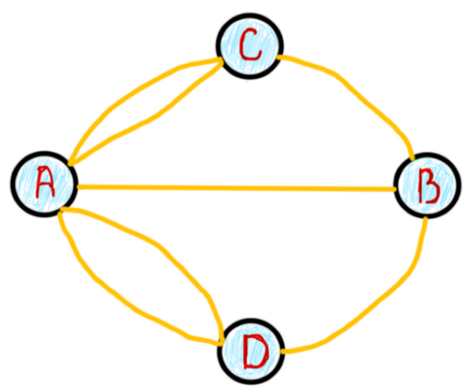


Seven Bridges of Königsberg

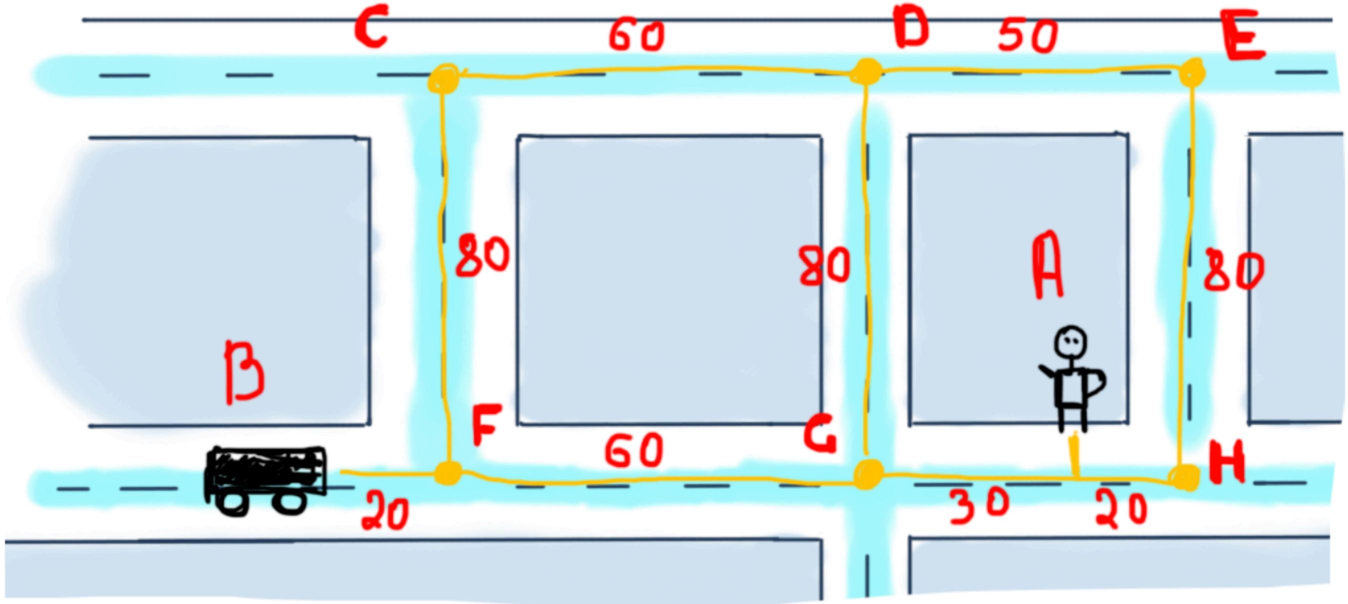


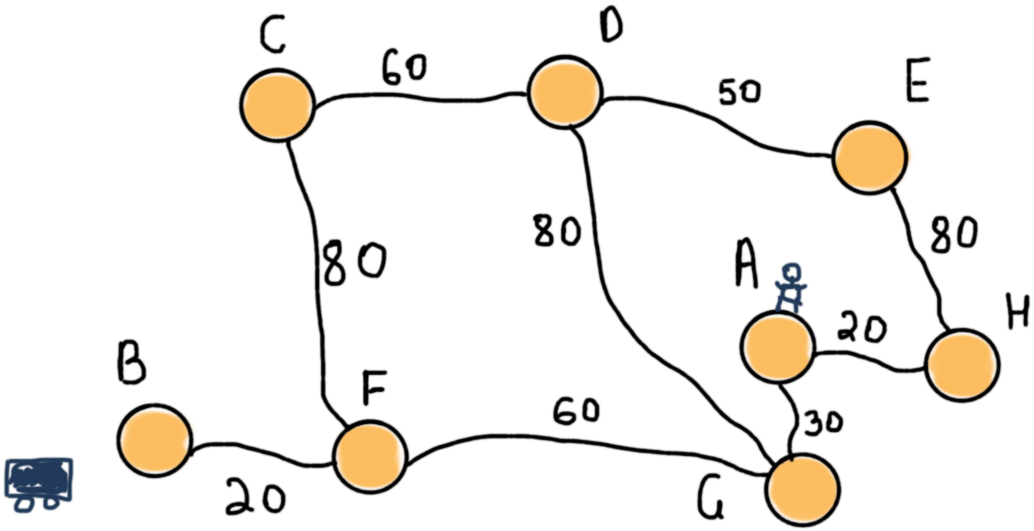




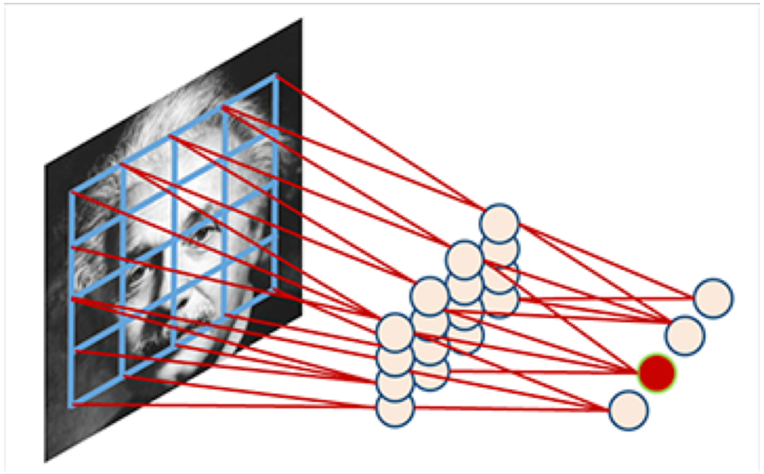


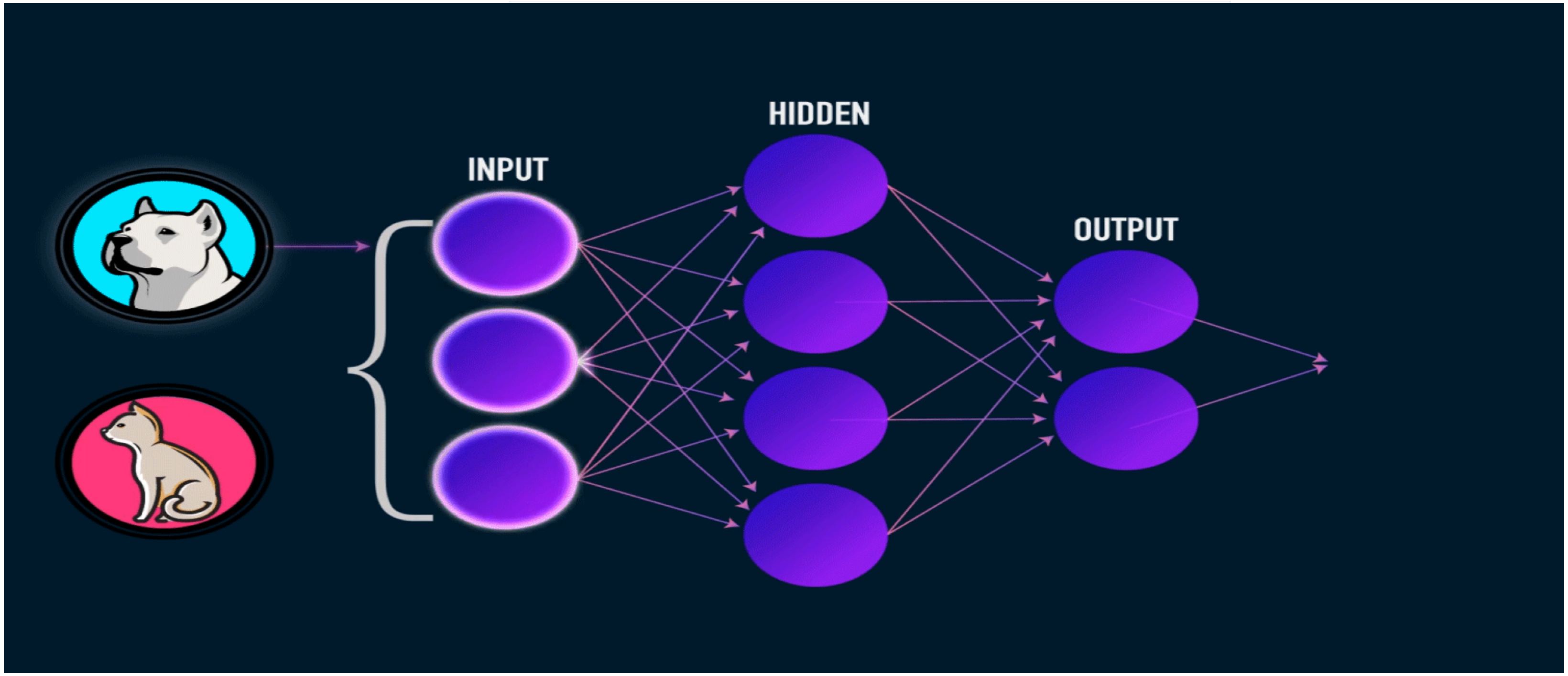
Efficient Route Choice



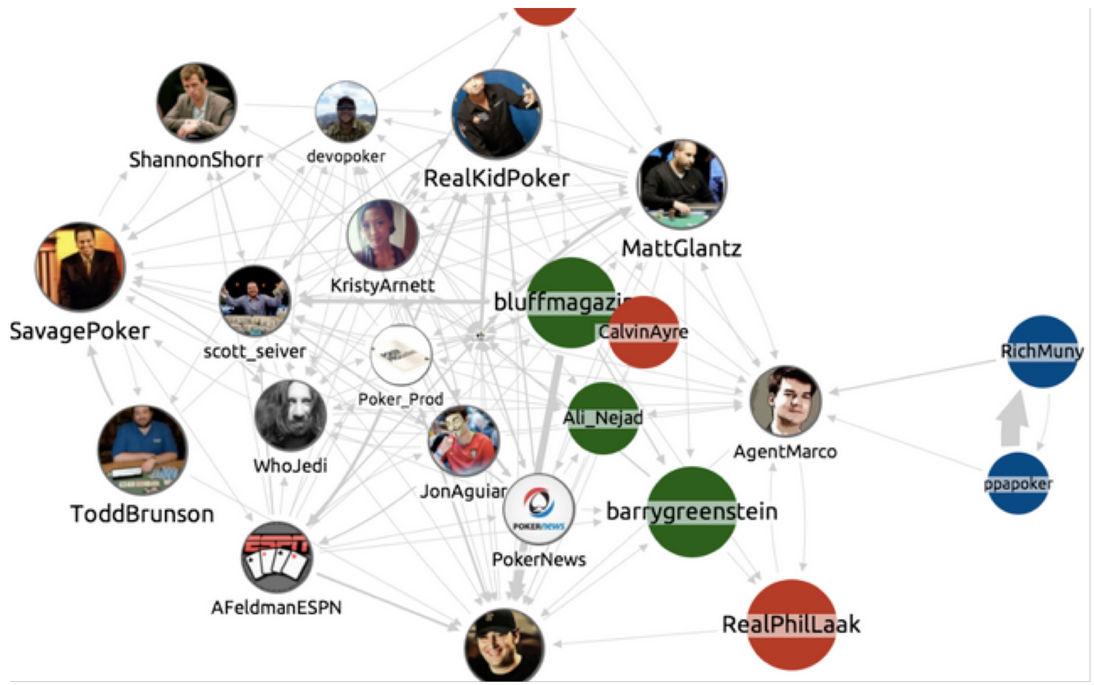


Artificial Neural Networks on Image

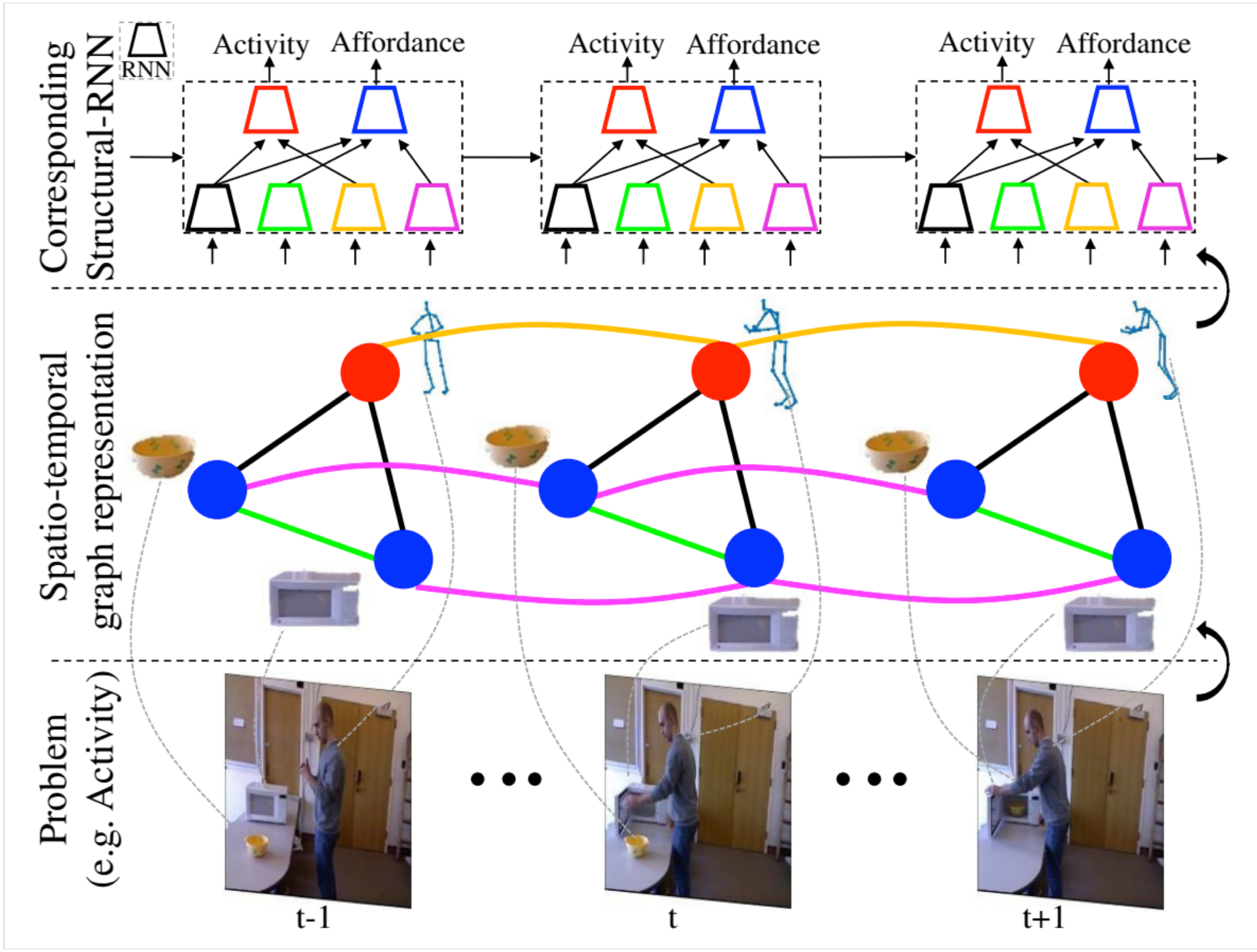




Social Media

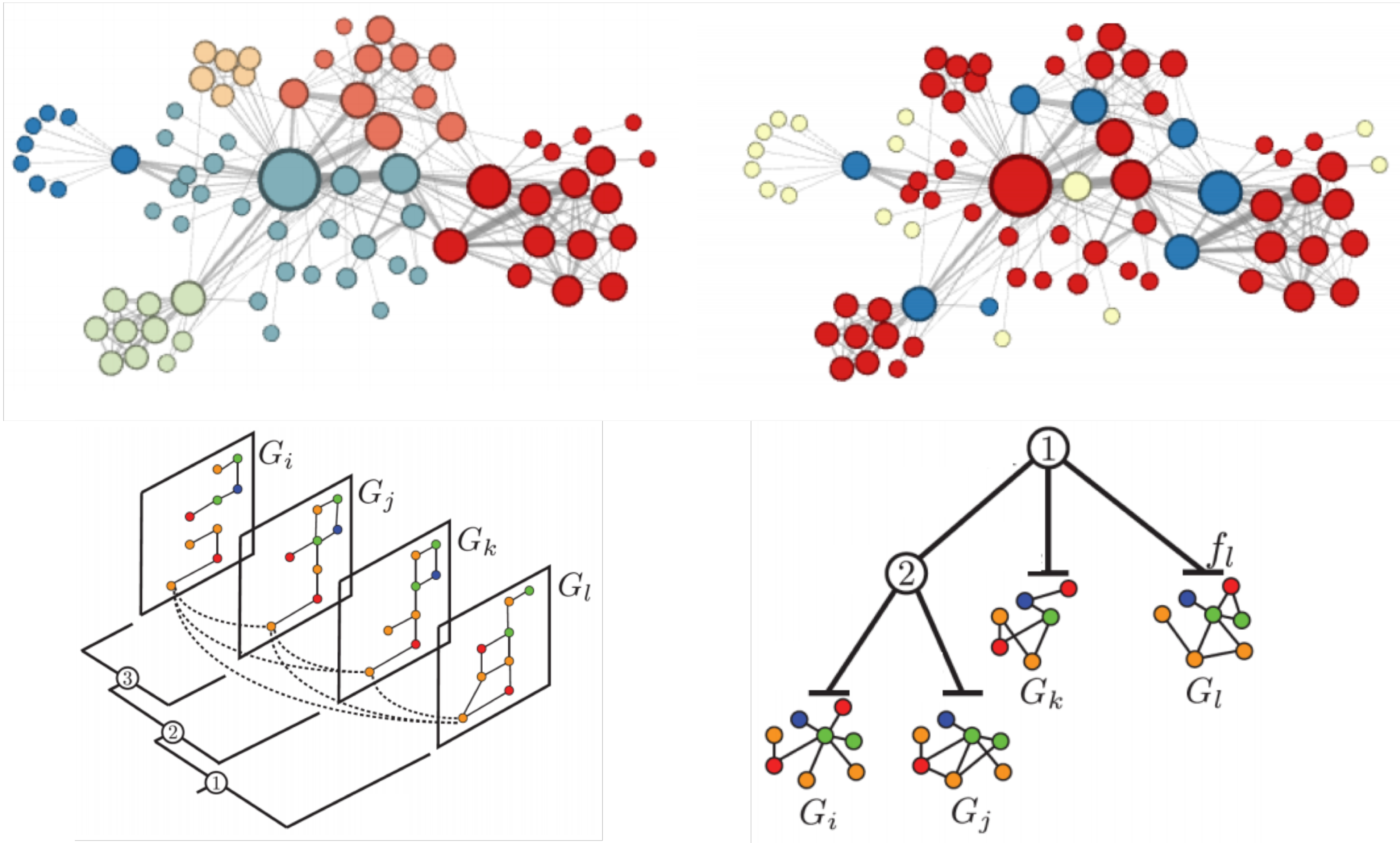


Graph Representation



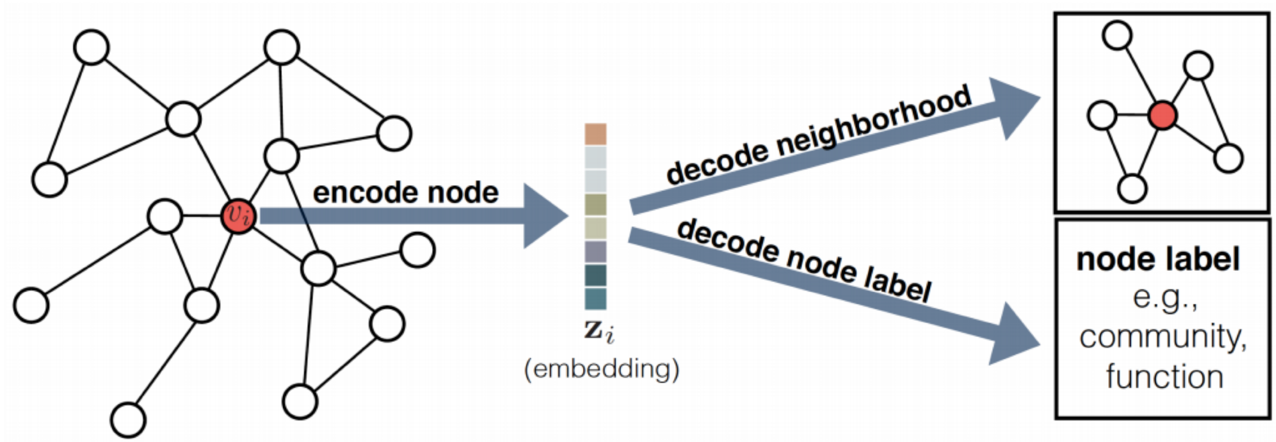
Ashesh Jain et al. “Structural-RNN: Deep Learning on Spatio-Temporal Graphs”, CVPR 2016.

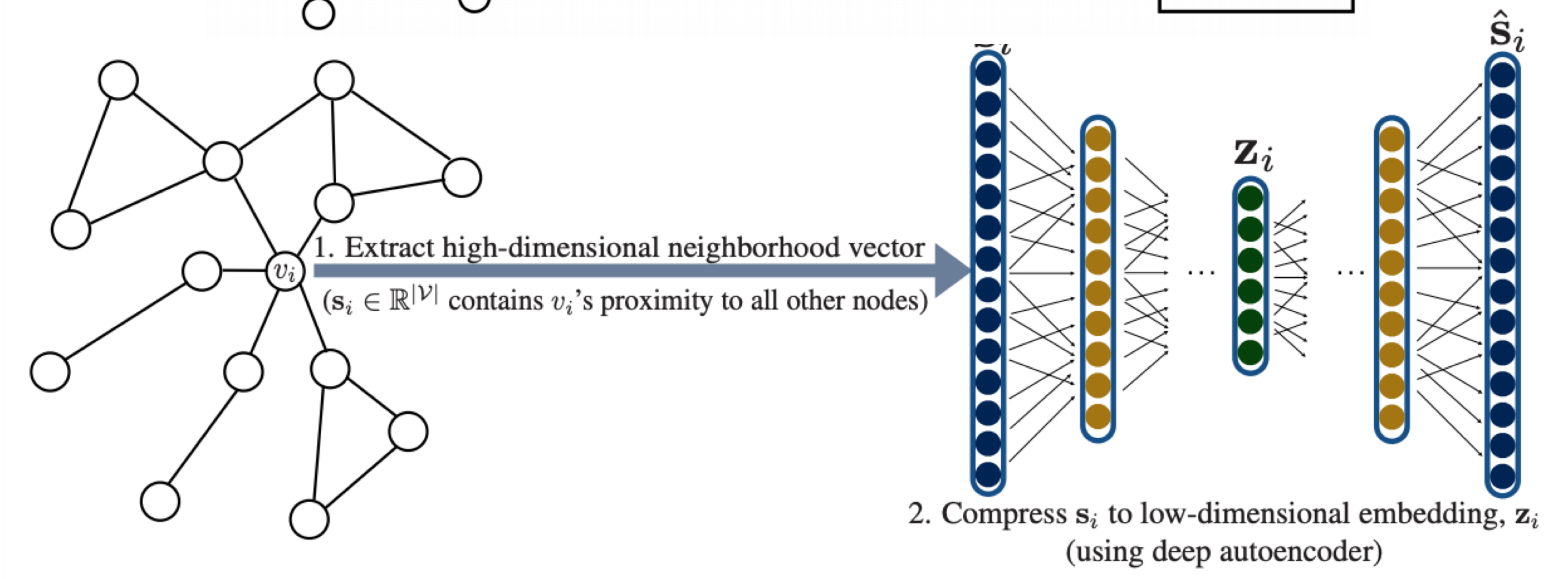
Representation Learning on Graphs



William L. Hamilton, et al. Representation Learning on Graphs: Methods and Applications. 2017.

Representation Learning on Graphs





William L. Hamilton, et al. Representation Learning on Graphs: Methods and Applications. 2017.

Next class

- Topic: Trees
- Pre-class reading: Chap 11