Lecture 12: Matrices
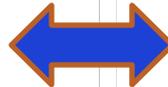
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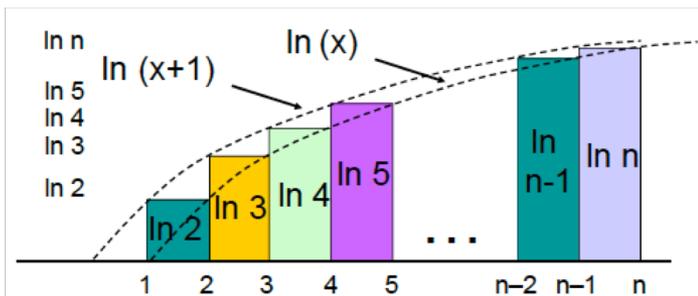
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Recap Previous Lecture

- Dealing with products, application of summation
- Cardinality of set, equal cardinality
- Countable set and uncountable set

$$\prod_{i=1}^{n} a_i := a_1 \cdot a_2 \cdots a_n$$

$$\ln(n!) = \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n)$$
$$= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n)$$
$$= \sum_{i=1}^{n} \ln(i)$$

| Finite Sets | Countably Infinite Sets | Uncountable Infinite Sets |

$$\int_{1}^{n} \ln(x)\, dx \le \sum_{i=1}^{n} \ln(i) \le \int_{0}^{n} \ln(x+1)\, dx$$

S is finite $A \subset S$, → $|A| < |S|$.

A and B are infinite sets and $A \subseteq B$, → $|A| = |B|$.

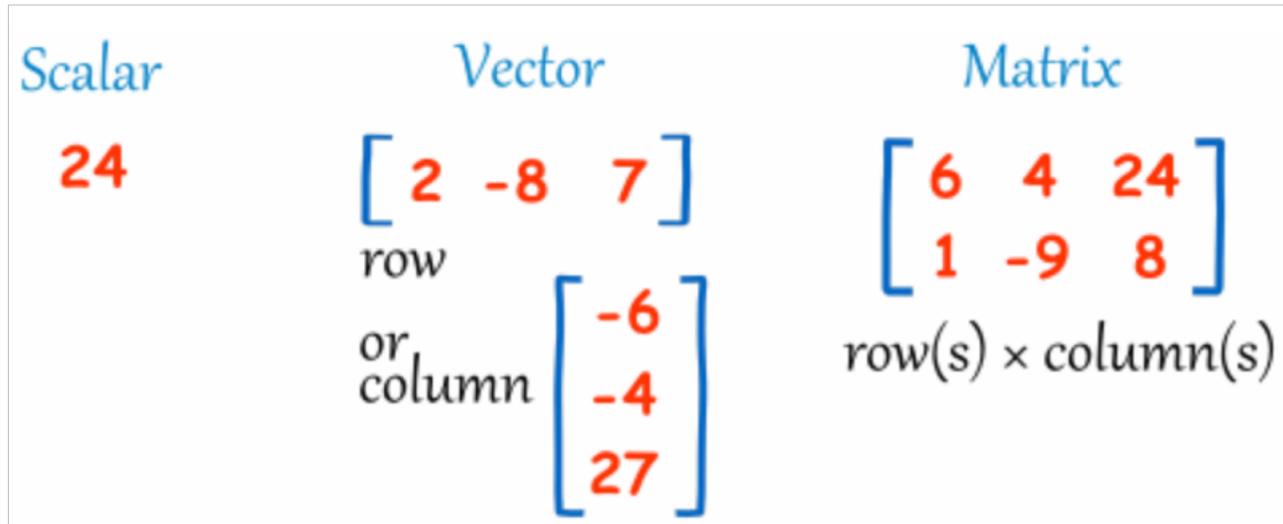A and B are countable sets, → $A \cup B$ is countable.

Outline

- Introduction to Matrix
- Matrix Arithmetic
- Zero-One Matrices

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- **Introduction to Matrix**
- Matrix Arithmetic
- Zero-One Matrices

Introduction

Scalar | Vector | Matrix
24

$$\begin{bmatrix} 2 & -8 & 7 \end{bmatrix}$$
row

or column
$$\begin{bmatrix} -6 \\ -4 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

row(s) × column(s)

- Scalars: A single number
- Vector: A 1D array of numbers, where each element is identified by an single index
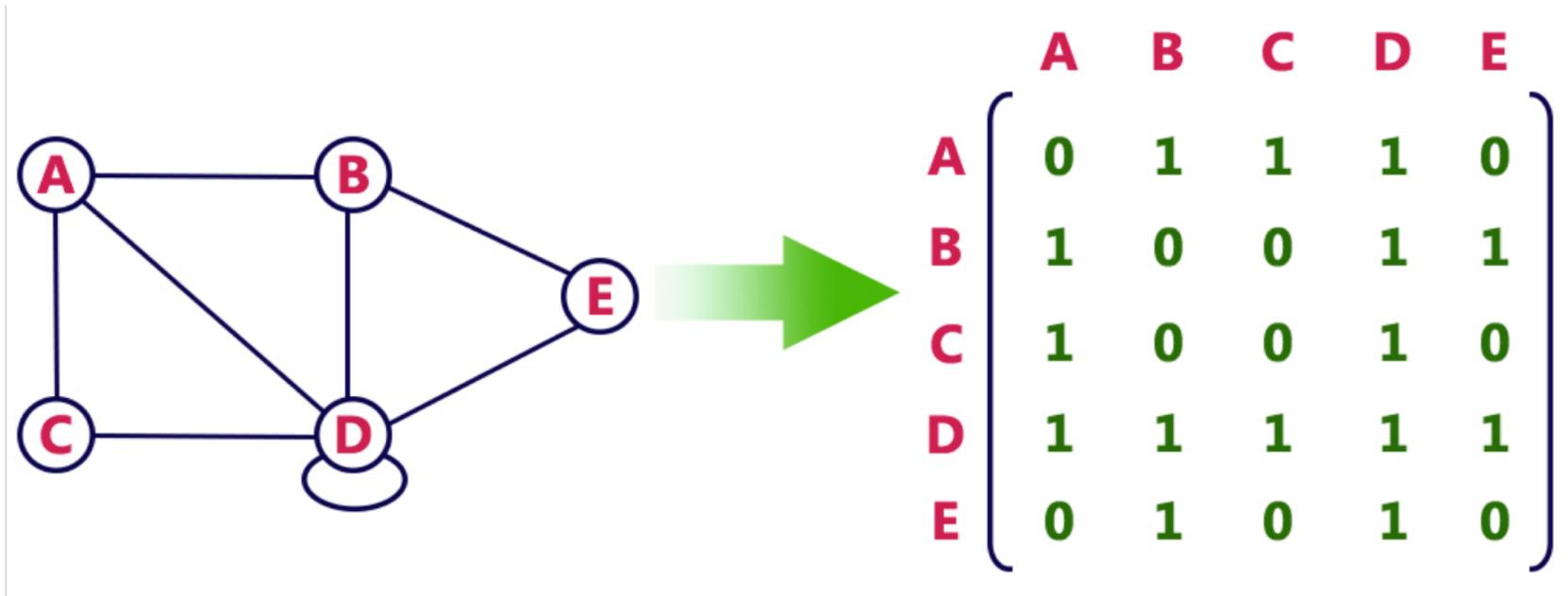- Matrix: A 2D array of numbers

Matrix

- Matrices are useful discrete structures that can be used in many ways. For example, they are used to:
    - describe certain types of functions known as linear transformations.
    - Express which vertices of a graph are connected by edges.

Linear transformation
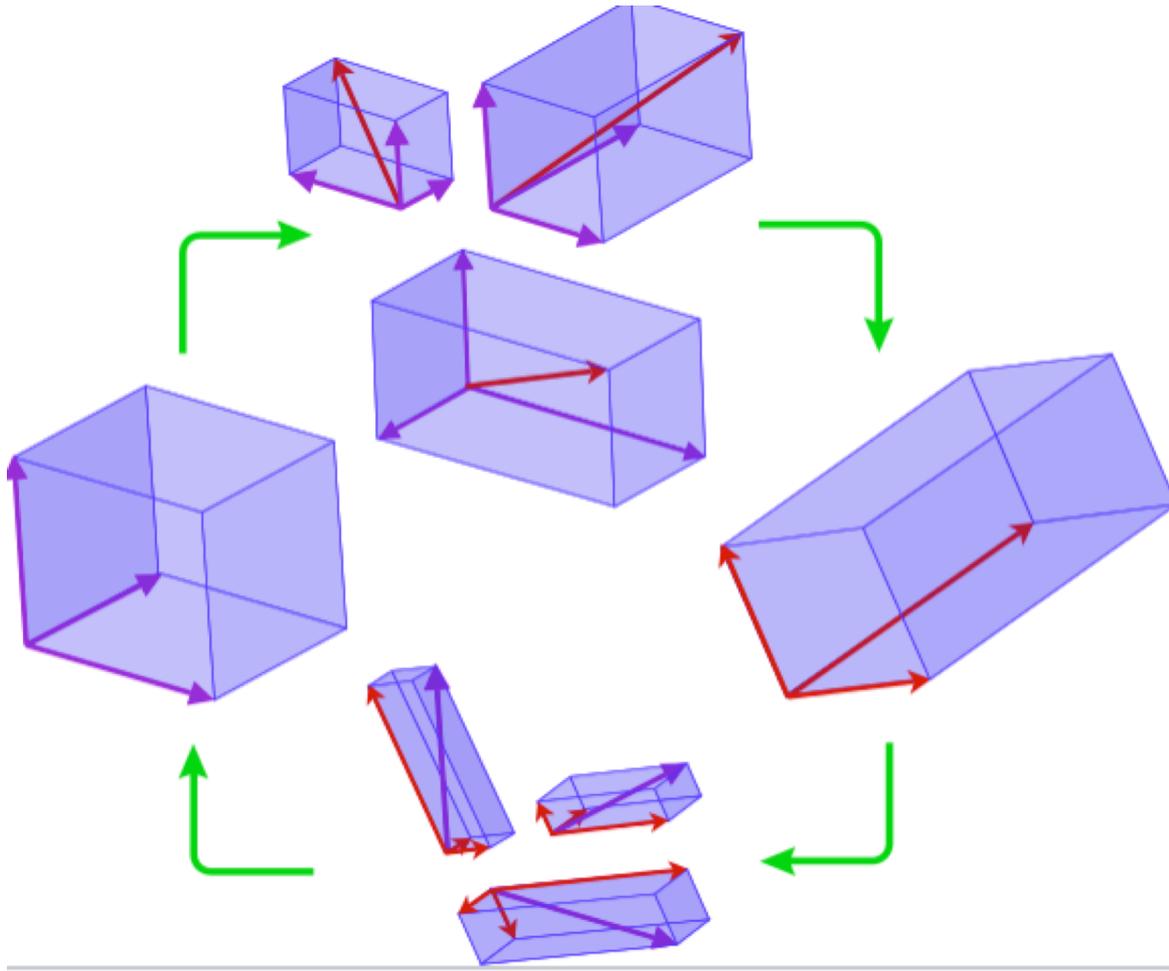
Matrix: a graph are connected by edges

Matrix

Definition: A *matrix* is a rectangular array of numbers. A matrix with *m* rows and *n* columns is called an $m \times n$ matrix.

- The plural of matrix is *matrices*.
- A matrix with the same number of rows as columns is called *square*.
- Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

3×2 matrix

3 by 2 matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

Notation

- Let m and n be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

- The i-th row of **A** is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \ldots, a_{in}]$. The j-th column of **A** is the $m \times 1$ *matrix:*

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix}$$

- The (i,j)-th *element* or *entry* of **A** is the element a_{ij}. We can use **A** = $[a_{ij}]$ to denote the matrix with its (i,j)-th element equal to a_{ij}.

Types of Matrices

- **Rectangular matrix**

- Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$m \neq n$$

Types of Matrices

- **Square matrix**

- The number of rows is equal to the number of columns

   (a square matrix   **A**   has an order of m)

m x m

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i=j$

Types of Matrices

- **Diagonal matrix**

- A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

$a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 0$ for some or all $i = j$

Types of Matrices

- **Unit or Identity matrix - I**

- A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

$a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 1$ for some or all $i = j$

Types of Matrices

- **Null (zero) matrix - 0**

- All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$a_{ij} = 0$ for all i, j

Types of Matrices

- **Triangular matrix**

- A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Types of Matrices

- **Upper Triangular matrix**

- A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$a_{ij} = 0$ for all $i > j$

Types of Matrices

- **Lower Triangular matrix**

- A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

$a_{ij} = 0$ for all $i < j$

Types of Matrices

- **Scalar matrix**

- A diagonal matrix whose main diagonal elements are equal to the same scalar

- A scalar is defined as a single number or constant

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$a_{ij} = 0$ for all $i \neq j$

$a_{ij} = a$ for some or all $i = j$

Outline

- Introduction to Matrix
- **Matrix Arithmetic**
- Zero-One Matrices

Matrix addition

Defintion: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The sum of **A** and **B**, denoted by **A** + **B**, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i,j)-th element. In other words, **A** + **B** = $[a_{ij} + b_{ij}]$.

Example:

$$
\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix}
+
\begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix}
=
\begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}
$$

Note that matrices of different sizes can not be added.

Matrix multiplication

Definition: Let **A** be an $n \times k$ matrix and **B** be a $k \times n$ matrix. The *product* of **A** and **B**, denoted by **AB**, is the $m \times n$ matrix that has its (i,j)-th element equal to the sum of the products of the corresponding elments from the i-th row of **A** and the j-th column of **B**. In other words, if **AB** = $[c_{ij}]$ then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{kj}b_{2j}$.

Example:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

The product of two matrices is undefined when the number of columns in the first matrix is not the same as the number of rows in the second.

Illustration of matrix multiplication

- The Product of **A** = [a_{ij}] and **B** = [b_{ij}]

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ . & . & & . \\ . & . & & . \\ a_{i1} & a_{i2} & \ldots & a_{ik} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \ldots & a_{mk} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & a_{12} & \ldots & b_{1j} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2j} & \ldots & b_{2n} \\ . & . & & . & & \\ . & . & & . & & \\ b_{k1} & b_{k2} & \ldots & b_{kj} & \ldots & b_{kn} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ . & . & & . \\ . & . & c_{ij} & . \\ . & . & & . \\ c_{m1} & c_{m2} & \ldots & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Matrix multiplication is not commutative

Example: Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Does **AB** = **BA**?

Solution:

$$\mathbf{AB} = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \qquad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

AB ≠ BA

Identity matrix and powers of matrices

Definition: The *identity matrix of order n* is the n x n matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$\mathbf{I_n} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

$\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$**,** when $\mathbf{A}$ is an $m \times n$ matrix

Powers of square matrices can be defined. When A is an $n \times n$ matrix, we have:

$$\mathbf{A}^0 = \mathbf{I}_n \qquad \mathbf{A}^r = \mathbf{AAA}\cdots\mathbf{A}$$

Transposes of matrices

Definition: Let $\mathbf{A} = [a_{ij}]$ be an $m{\times}n$ matrix. The *transpose* of $\mathbf{A}$, denoted by $\mathbf{A}^{\mathrm{T}}$, is the $n{\times}m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$.

If $\mathbf{A}^{\mathrm{T}} = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Transposes of matrices

Definition: A square matrix **A** is called symmetric if $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$. Thus $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is square.

Square matrices do not change when their rows and columns are interchanged.

Outline

- Introduction to Matrix

- Matrix Arithmetic

- **Zero-One Matrices**

Zero-one matrices

Definition: A matrix all of whose entries are either 0 or 1 is called a *zero-one matrix*

Algorithms operating on discrete structures represented by zero-one matrices are based on Boolean arithmetic defined by the following Boolean operations:

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Zero-one matrices

Definition: Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be an $m \times n$ zero-one matrices.

- The *join* of $\mathbf{A}$ and $\mathbf{B}$ is the zero-one matrix with (i,j)-th entry $a_{ij} \vee b_{ij}$. The *join* of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \vee \mathbf{B}$.

- The meet of of $\mathbf{A}$ and $\mathbf{B}$ is the zero-one matrix with (i,j)-th entry $a_{ij} \wedge b_{ij}$. The *meet* of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \wedge \mathbf{B}$.

Joins and meets of zero-one matrices

Example: Find the join and meet of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution:

The join of **A** and **B** is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of **A** and **B** is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Boolean product of zero-one matrices

Definition: Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero-one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero-one matrix. The *Boolean product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ zero-one matrix with (i,j)-th entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

Example: Find the Boolean product of $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Boolean product of zero-one matrices

Solution: The Boolean product **A** $\odot$ **B** is given by

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \qquad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Boolean product of zero-one matrices

Definition: Let **A** be a square zero-one matrix and let *r* be a positive integer. The *r*-th Boolean power of **A** is the Boolean product of *r* factors of **A**, denoted by $\mathbf{A}^{[r]}$. Hence,

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot ... \odot \mathbf{A}}_{r \text{ times}}.$$

We define $\mathbf{A}^{[r]}$ to be $\mathbf{I}_n$.

(The Boolean product is well defined because the Boolean product of matrices is associative.)

Boolean product of zero-one matrices

Example: Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

Find $\mathbf{A}^n$ for all positive integers n.

Solution:

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^{[\mathbf{n}]} = \mathbf{A}^{\mathbf{5}} \quad \text{for all positive integers } n \text{ with } n \geq 5.$$

Next class

- Topic: Algorithm
- Pre-class reading: Chap 3.1