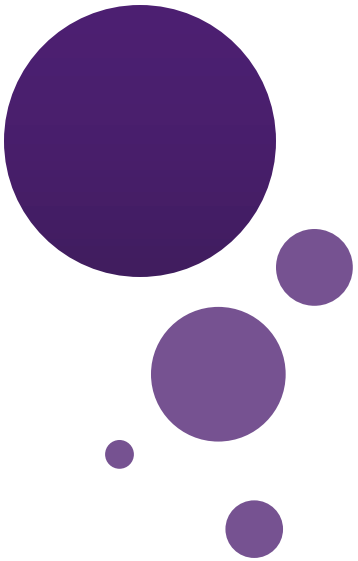

Lecture 17: Modular Arithmetic

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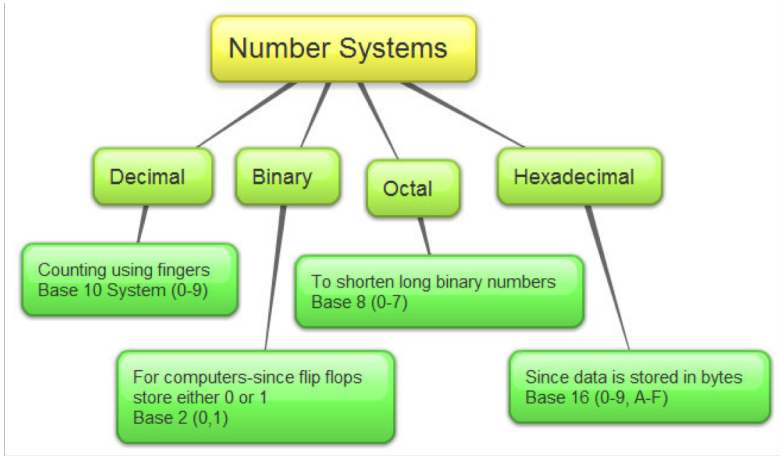
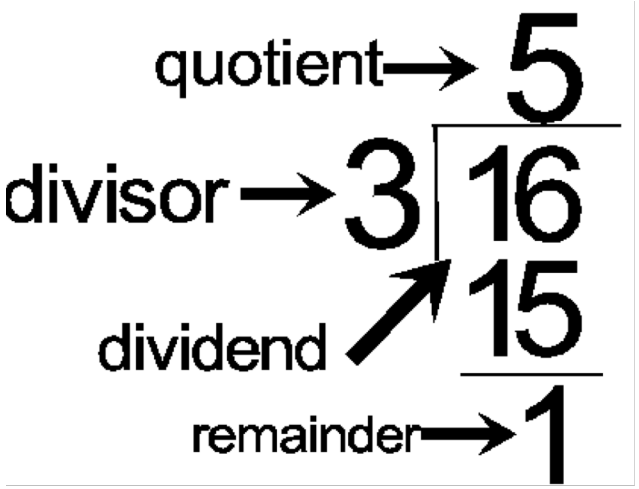
Recap Previous Lecture

- Representation of Integer
- Division algorithm, properties of division.
- Division of negative integer.





Outline

- Introduction to Modular Arithmetic
- Properties of Mod Function
- Arithmetic Modulo m
- Applications

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Division Algorithm – Theorem of Existence

- *Recall:*
- According to the Division Algorithm
  - 32 mod 7 = 4 because 32 = 7·4 + 4.
  - −10 mod 7 = 4 because −10 = 7·(-2) + 4.
- What is common between 32 and -10?

  The remainder.

Modulus

- The concept of positive remainder allows mapping of any set of integers to its subset defined by any positive integer *m* (modulus)
  - f: $S \subseteq \mathbf{Z} \rightarrow \mathbf{Z}_m = \{0, \ldots, m\text{-}1\} \subseteq S$.

Modular Arithmetic

- Let $(a, b) \in Z^2$, $m \in Z^+$ then a is a congruent to b modulo m if m divides a – b.

  Notation: $a \equiv b \pmod{m}$.

- Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if a **mod** m = b **mod** m.

- The equivalent definitions of congruency:

Let m be a positive integer. The integers a and b are congruent modulo m if and only if

$$\exists k \in Z;\ a = b + km$$

(mod *m*) and mod *m*

- *a* mod *m* = *b* denotes function that results in *b*.
- *a* ≡ *b* (mod *m*) denotes statement ("equation") that could be true or false.

Example

- $17 \equiv 5 \pmod 6$?

  $6 | (17 - 5) = 12 \Rightarrow 17 \equiv 5 \pmod 6$

- $24 \equiv 14 \pmod 6$?

  6 does not divide 10

  $\Rightarrow$ 24 is not congruent to 14 (mod 6)

Algebraic Operations and Congruencies

- If $a \equiv b \pmod{m}$ holds then $c \cdot a \equiv c \cdot b \pmod{m}$, where c is any integer, holds.
- If $a \equiv b \pmod{m}$ holds then $c + a \equiv c + b \pmod{m}$, where c is any integer, holds.
- Dividing a congruence by an integer does not always produce a valid congruence.
- **Example**:
- The congruence $14 \equiv 8 \pmod{6}$ holds.
- But dividing both sides by 2 does not produce a valid congruence since $14/2 = 7$ and $8/2 = 4$, but $7 \not\equiv 4 \pmod{6}$.

Example

- $17 \equiv 5 \pmod 6$

  $6 | (17 - 5) = 12 \Rightarrow 17 \equiv 5 \pmod 6$

- $8 \equiv 14 \pmod 6$

  $6 | (8-14) = -6 \Rightarrow 8 \equiv 14 \pmod 6$

$17 + 8 \equiv 5 + 14 \pmod 6$? $\Rightarrow$ $25 \equiv 19 \pmod 6$?

$17 \times 8 \equiv 5 \times 14 \pmod 6$? $\Rightarrow$ $136 \equiv 70 \pmod 6$?

Congruencies of Sums and Products

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $a \equiv b \pmod{m}$, then

$a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$

Example:

- Because $7 \equiv 2 \pmod{5}$ and $11 \equiv 1 \pmod{5}$,

  $18 = 7 + 11 \equiv 2 + 1 = 3 \pmod{5}$

- $77 = 7 \cdot 11 \equiv 2 \cdot 1 = 2 \pmod{5}$

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Properties of **mod** Function

- Let *m* be a positive integer and let *a* and *b* be integers.
- Then
- $(a + b) \pmod m = ((a \bmod m) + (b \bmod m)) \bmod m$

and

- $ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$.

Practice

- It is often asked to find remainder of the positive powers of 10 over an integer number:

  $10 \bmod 3$, $10^2 \bmod 3$, $10^3 \bmod 3$, etc.

- or

  $10 \bmod 7$, $10^2 \bmod 7$, $10^3 \bmod 7$, etc.

Practice

- Use of the properties of the mod function reduce computing:
  - $10^n \bmod m = (10 \bmod m)^n \bmod m$, where $n \in \mathbf{Z}^+$.

- $10 \bmod 3 = 1 \rightarrow 10^n \bmod 3 = (10 \bmod 3)^n \bmod 3 = 1$ (Division Algorithm: $1 = 0 \times 3 + 1$)
- $10 \bmod 7 = 3 \rightarrow 10^n \bmod 7 = (10 \bmod 7)^n \bmod 7 = 3^n \bmod 7$

Computations

- Suppose we need to compute a^n mod m. It could be solved by sequences of multiplications and divisions. There is the faster approach.
- For example, we need to compute a^8 mod n. Instead of 7 multiplications and one reduction mod of a big number perform 3 multiplications and 3 mod operations of a smaller numbers:
    - a^8 mod m = $((a^2 \bmod m)^2 \bmod m)^2 \bmod m$.
- The same,
    - a^{16} mod m = $(((a^2 \bmod m)^2 \bmod m)^2 \bmod m)^2 \bmod m$.

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Arithmetic Modulo *m*

Definitions: Let $\mathbf{Z}_m$ be the set of nonnegative integers less than m: {0,1, …., $m-1$}

- The operation $+_m$ is defined as $a +_m b = (a + b)$ **mod** m. This is *addition modulo m*.
- The operation $\cdot_m$ is defined as $a \cdot_m b = (a \cdot b)$ **mod** m. This is *multiplication modulo m*.
- Using these operations is said to be doing *arithmetic modulo m*.

Example

- **Example**: Find $7 +_{11} 9$ and $7 \cdot_{11} 9$.

  Solution: Using the definitions above:

  - $7 +_{11} 9 = (7 + 9) \ \mathbf{mod}\ 11 = 16 \ \mathbf{mod}\ 11 = 5$
  - $7 \cdot_{11} 9 = (7 \cdot 9) \ \mathbf{mod}\ 11 = 63 \ \mathbf{mod}\ 11 = 8$

Properties of Arithmetic Modulo *m*

- The operations $+_m$ and $\cdot_m$ satisfy many of the same properties as ordinary addition and multiplication if operands belong to $\mathbf{Z}_m$.
  - *Closure*: If a and b belong to $\mathbf{Z}_m$, then $a +_m b$ and $a \cdot_m b$ belong to $\mathbf{Z}_m$.
  - *Associativity*: If a, b, and c belong to $\mathbf{Z}_m$, then $(a +_m b) +_m c = a +_m (b +_m c)$ and $(a \cdot_m b) \cdot_m c = a \cdot_m (b \cdot_m c)$.
  - *Commutativity*: If a and b belong to $\mathbf{Z}_m$, then $a +_m b = b +_m a$ and $a \cdot_m b = b \cdot_m a$.
  - *Identity elements*: The elements 0 and 1 are identity elements for addition and multiplication modulo m, respectively.
    - If a belongs to $\mathbf{Z}_m$, then $a +_m 0 = a$ and $a \cdot_m 1 = a$.

Properties of Arithmetic Modulo *m*, cont'd

- *Additive inverses*: If $a \neq 0$ belongs to $\mathbf{Z}_m$, then $m - a$ is the additive inverse of a modulo m and 0 is its own additive inverse.
  - $a +_m (m - a) = 0$ and $0 +_m 0 = 0$

- *Distributivity*: If *a*, *b,* and *c* belong to $\mathbf{Z}_m$, then
  - $a \cdot_m (b +_m c) = (a \cdot_m b) +_m (a \cdot_m c)$ and
    $(a +_m b) \cdot_m c = (a \cdot_m c) +_m (b \cdot_m c)$.

- Multiplicative inverses have not been included since they do not always exist. For example, there is no multiplicative inverse of 2 modulo 6

Practice

- **Problem:** Prove that A mod 3 = D mod 3, where A is a positive integer number, D is a sum of decimal digits of A.

- For example, 6371 mod 3 = 17 mod 3.

- **Solution:** Weighted Positional Notation $A = a_n 10^n + \ldots + a_1 10^1 + a_0 10^0$ (e.g. $6371 = 6 \times 10^3 + 3 \times 10^2 + 7 \times 10^1 + 1 \times 10^0$).
- $A \bmod 3 = (a_n \times 10^n + \ldots + a_1 \times 10^1 + a_0 \times 10^0) \bmod 3$

  $= ((a_n \times 10^n) \bmod 3 + \ldots + (a_1 \times 10^1) \bmod 3 + (a_0 \times 10^0) \bmod 3) \bmod 3$

Each term $(a_i \times 10^i) \bmod 3 = ((a_i \bmod 3) \times (10^i \bmod 3)) \bmod 3 = ((a_i \bmod 3) \times 1) \bmod 3$ (see Practice example above), i.e.

- $A \bmod 3 = ((a_n \bmod 3) \bmod 3 + \ldots + (a_1 \bmod 3) \bmod 3 + (a_0 \bmod 3) \bmod 3) \bmod 3 =$

  $= (a_n \bmod 3 + \ldots + a_1 \bmod 3 + a_0 \bmod 3) \bmod 3$

  $= (a_n + \ldots + a_1 + a_0) \bmod 3$

Results of Modular Arithmetic

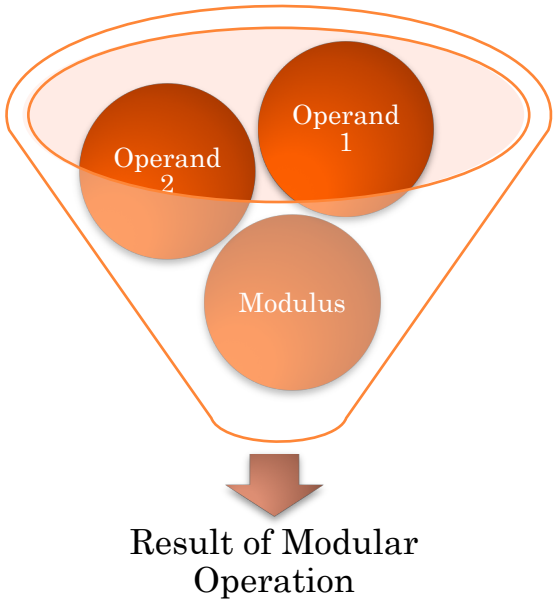
Module arithmetic replaces operations on integer numbers with operations on remainders of the division of the numbers by specified positive integer *m*.

It is obvious that different integers may result in the same numbers:

(16x2) mod 7 = 4

(20−30) mod 7 = 4

How to evaluate such results?



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Applications of Congruences: Hash Functions

Assignment of memory location to a student record

$$h(k) = k \textbf{ mod } m$$

Key: social security #

# of available memory location

Example: h (064212848) = 064212848 mod 111 = 14
when m = 111

Applications of Congruences: Cryptology

a) Encryption:

- Making messages secrets by shifting each letter three letters forward in the alphabet

  B → E X → A

- Mathematical expression:

  $f(p) = (p + 3) \bmod 26 \quad 0 \leq p \leq 25$

Applications of Congruences: Cryptology

- Example: What is the secret message produced from the message “Meet you in the park”

Solution:

1. Replace letters with numbers:
   meet = 12 4 4 19
   you = 24 14 20
   in = 8 1 3
   the = 19 7 4
   park = 15 0 17 10
2. Replace each of these numbers p by $f(p) = (p + 3) \bmod 26$
   meet = 15 7 7 22
   you = 1 17 23
   in = 11 16
   the = 22 10 7
   park = 18 3 20 13
3. Translate back into letters: “PHHW BRX LQ WKH SDUN”

Applications of Congruences: Cryptology

b) Decryption (Deciphering)

$f(p) = (p + k)$ **mod** 26 (shift cepher)
$\Rightarrow f^{-1}(p) = (p - k)$ **mod** 26

Caesar's method and shift cipher are very vulnerable and thus have low level of security (reason frequency of occurrence of letters in the message)
$\Rightarrow$ Replace letters with blocks of letters.

Next class

- Topic: Integer Representations
- Pre-class reading: Chap 4.2