UNIVERSITY AT ALBANY
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Lecture 20: Cryptography

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Recap Previous Lecture

- Prime Factorization
- GCD and LCM
- Euclidean Algorithm

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}, b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$$

$$\gcd(a,b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \cdots p_n^{\min(a_n,b_n)}$$

$$120 = 2^3 \cdot 3 \cdot 5, \quad 500 = 2^2 \cdot 5^3$$

$$\gcd(120,500) = 2^2 \cdot 3^0 \cdot 5^1 = 20$$

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}, b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$$

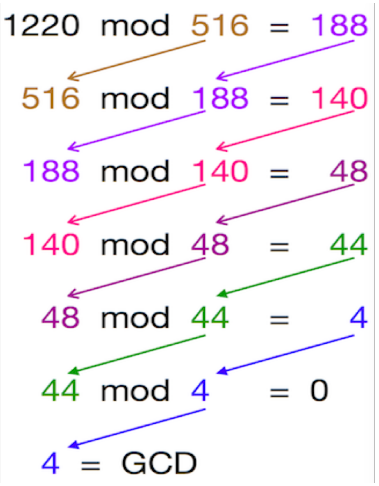
$$\operatorname{lcm}(a,b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \cdots p_n^{\max(a_n,b_n)}$$

$$120 = 2^3 \cdot 3 \cdot 5, 500 = 2^2 \cdot 5^3$$

$$\operatorname{lcm}(120,500) = 2^3 \cdot 3^1 \cdot 5^3 = 8 \cdot 3 \cdot 125 = 3000$$

$$\begin{aligned} r_0 &= r_1 q_1 + r_2, 0 \le r_2 < r_1 \\ r_1 &= r_2 q_2 + r_3, 0 \le r_3 < r_2 \\ &\ \ \cdots \\ r_{n-2} &= r_{n-1} q_{n-1} + r_n, 0 \le r_n < r_{n-1} \\ r_{n-1} &= r_n q_n \end{aligned}$$

$$\begin{aligned} \gcd(a,b) &= \gcd(r_0, r_1) = \gcd(r_1, r_2) = \cdots = \gcd(r_{n-2}, r_{n-1}) \\ &= \gcd(r_{n-1}, r_n) = \gcd(r_n, 0) = r_n \end{aligned}$$



1220 mod 516 = 188

516 mod 188 = 140

188 mod 140 = 48

140 mod 48 = 44

48 mod 44 = 4

44 mod 4 = 0

4 = GCD

Outline

- Introduction to Cryptograph
- Turing Code
- Public Key Cryptography
- RSA Cryptosystem

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- **Introduction to Cryptograph**
- Turing Code
- Public Key Cryptography
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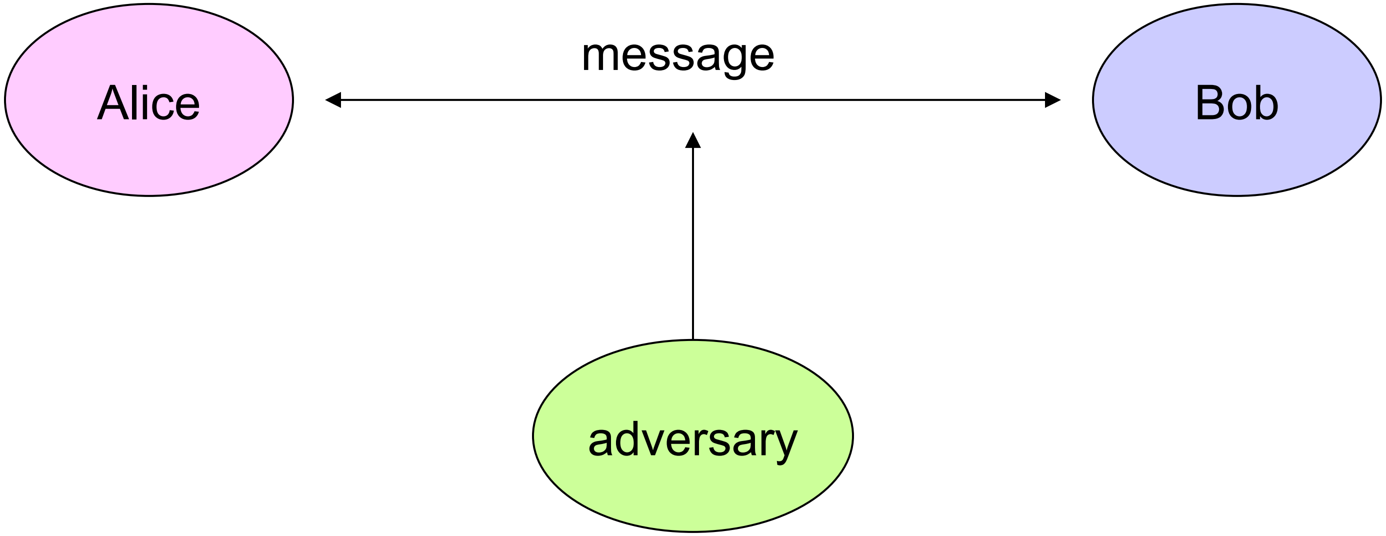
Cryptograph

Cryptography is the study of methods for sending and receiving secret messages.



Goal: Even though an adversary can listen to your conversation, the adversary can not learn what the message was.

Cryptograph

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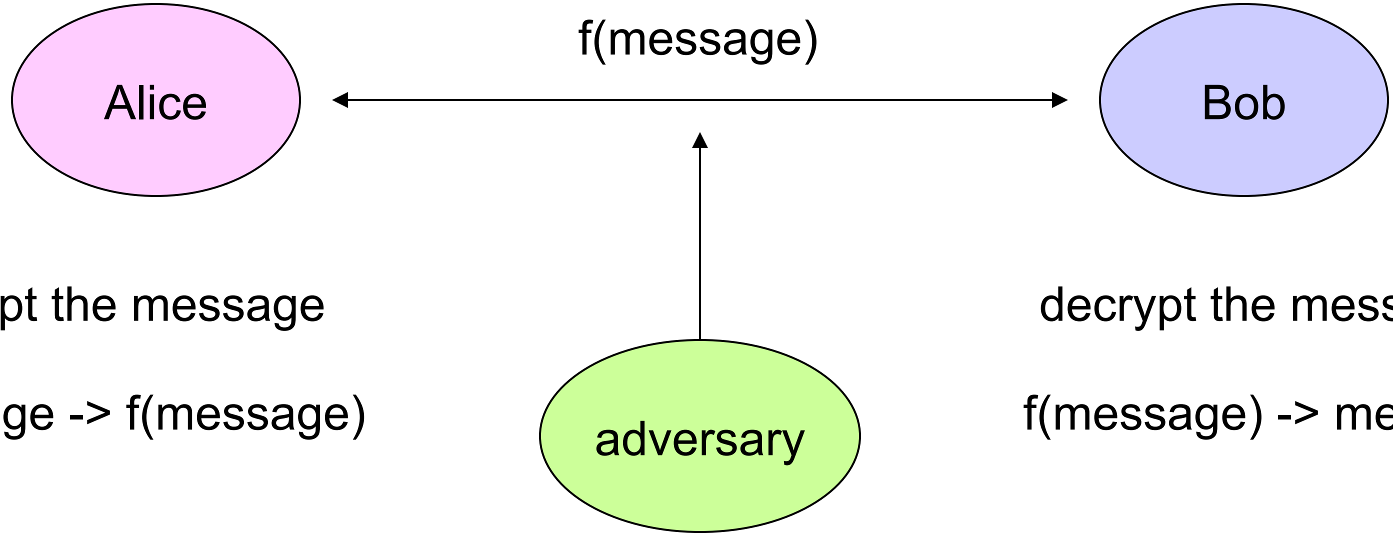


encrypt the message

message -> f(message)

decrypt the message

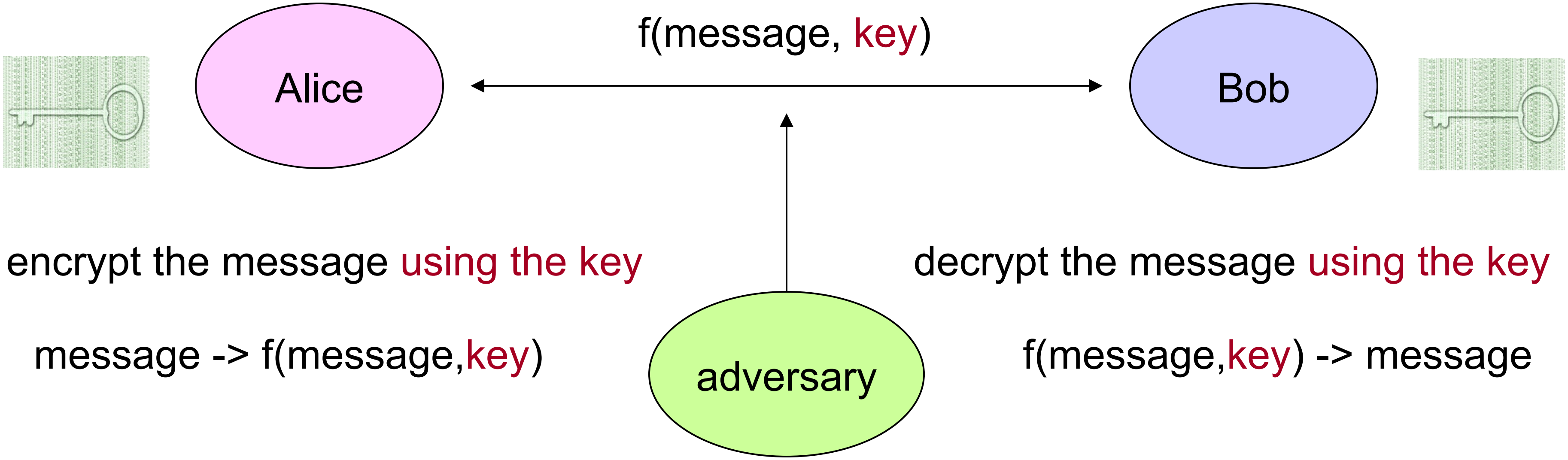
f(message) -> message

But the adversary has no clue how to obtain message from f(message)

A difficult goal!

Cryptograph: Key

Goal: Even though an adversary can listen to your conversation, the adversary can not learn what the message was.



f(message, key)

Alice

Bob

adversary

encrypt the message using the key

message -> f(message,key)

decrypt the message using the key

f(message,key) -> message

But the adversary can not decrypt f(message,key) without the key

Use number theory!

Outline

- Introduction to Cryptograph
- **Turing Code**
- Public Key Cryptography
- RSA Cryptosystem

Turing’s Code (Version 1.0)

The first step is to translate a message into a number

“v i c t o r y”

-> 22 09 03 20 15 18 25

Beforehand The sender and receiver agree on a secret key, which is a large number k.

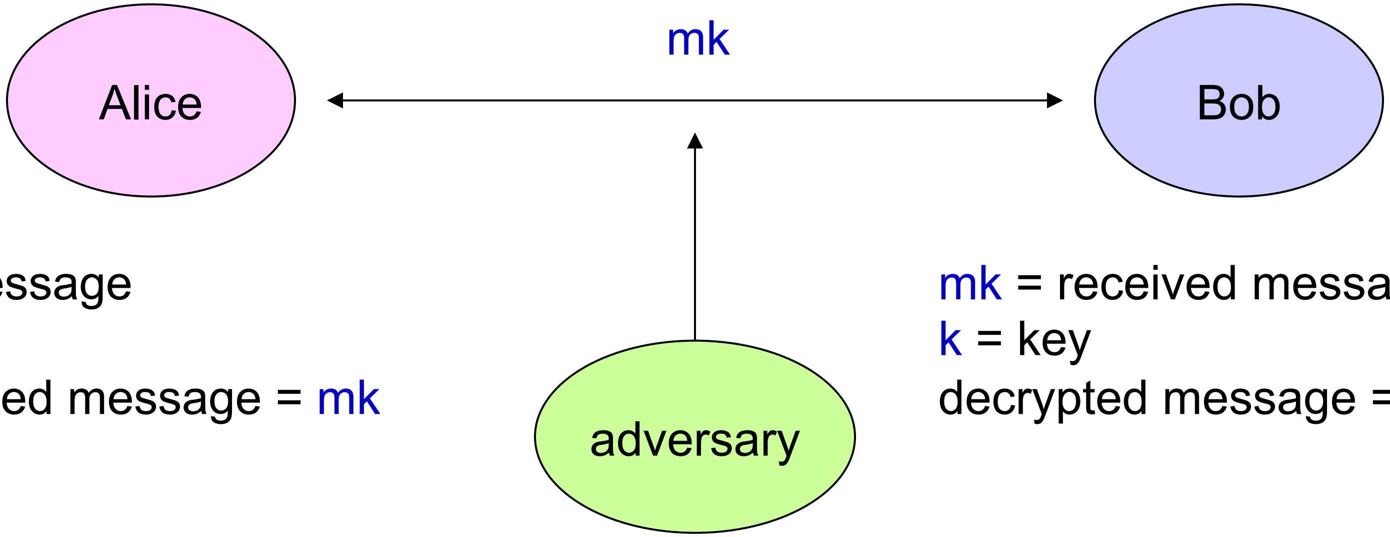
Encryption The sender encrypts the message m by computing:

$$m^* = m \cdot k$$

Decryption The receiver decrypts m by computing:

$$m^*/k = m \cdot k/k = m$$

Turing’s Code (Version 1.0)



m = message
k = key
encrypted message = mk

mk = received message
k = key
decrypted message = mk/k=m

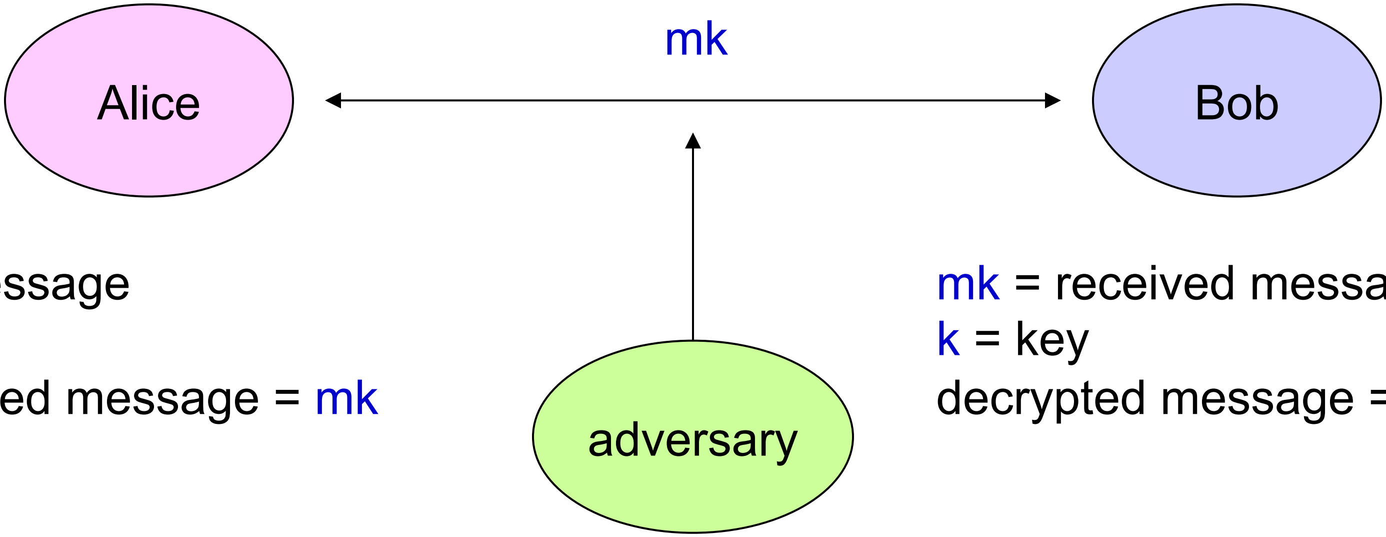
Why the adversary cannot figure out m?

The adversary doesn’t have the key k,

and so can only factor mk to figure out m,

but factoring is a difficult task to do.

Turing's Code (Version 1.0)



m = message
k = key
encrypted message = mk

mk = received message
k = key
decrypted message = mk/k=m

So why don't we use this Turing's code today?

Major flaw: if you use the same key to send two messages m and m',
then from mk and m'k,
we can use gcd(mk,m'k) to figure out k,
and then decrypt every message.

Turing’s Code (Version 2.0)

Beforehand The sender and receiver agree on a large prime p, which may be made public. (This will be the modulus for all our arithmetic.) They also agree on a secret key k in $\{1, 2, . . . , p − 1\}$.

Encryption The message m can be any integer in the set $\{0, 1, 2, . . . , p − 1\}$. The sender encrypts the message m to produce m^* by computing:

$$m^* = mk \bmod p$$

Decryption Let k' be the multiplicative inverse of k under modulo p.

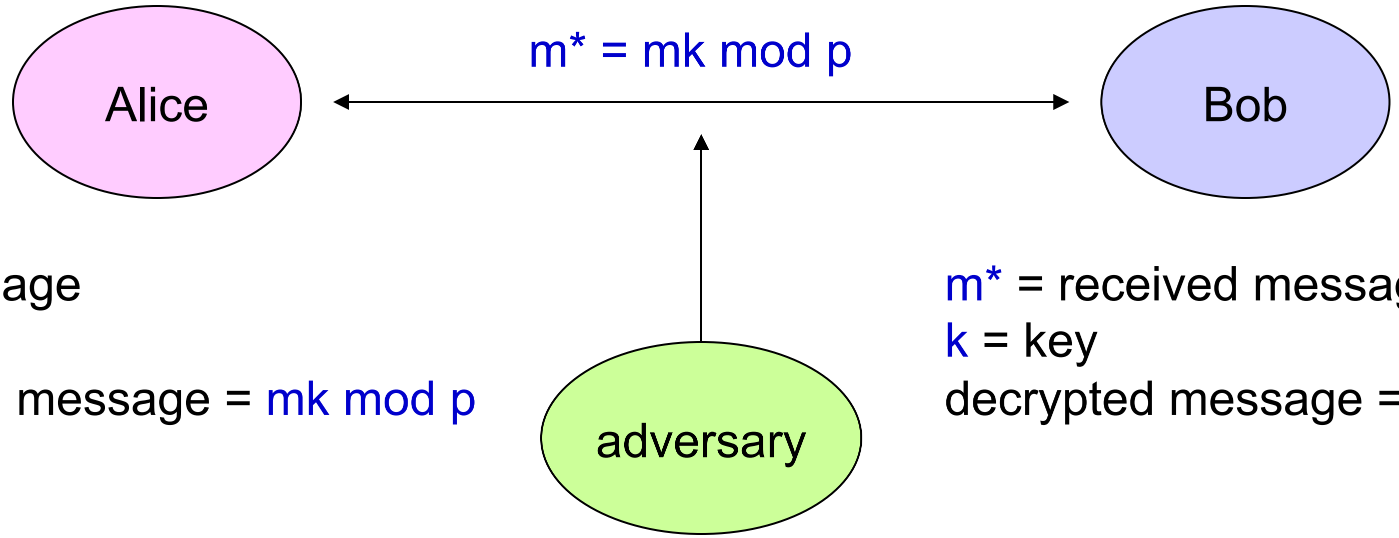
$$m^* \equiv mk \pmod{p}$$

$$m^*k' \equiv m \pmod{p}$$

$$m^*k' \equiv m$$

Turing’s Code (Version 2.0)

Public information p



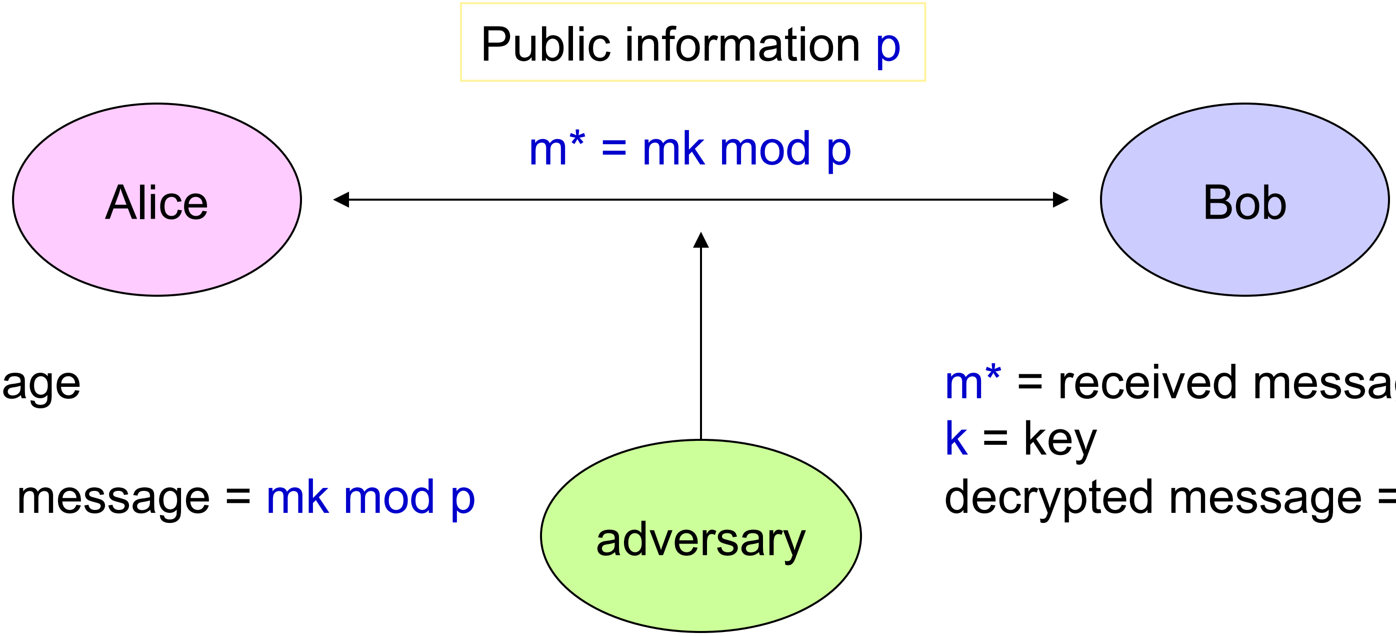
m = message
k = key
encrypted message = mk mod p

m* = received message
k = key
decrypted message = m*k’ =m

Why the adversary cannot figure out m?

Many m and k can produce m* as output,
just impossible to determine m without k.

Turing’s Code (Version 2.0)



Public information p

m* = mk mod p

Alice

Bob

adversary

m = message
k = key
encrypted message = mk mod p

m* = received message
k = key
decrypted message = m*k’ =m

So why don’t we use this Turing’s code today?

plain-text attack

If the adversary somehow knows m,
then first compute m’ := multiplicative inverse of m

$m^* \equiv mk \pmod{p}$

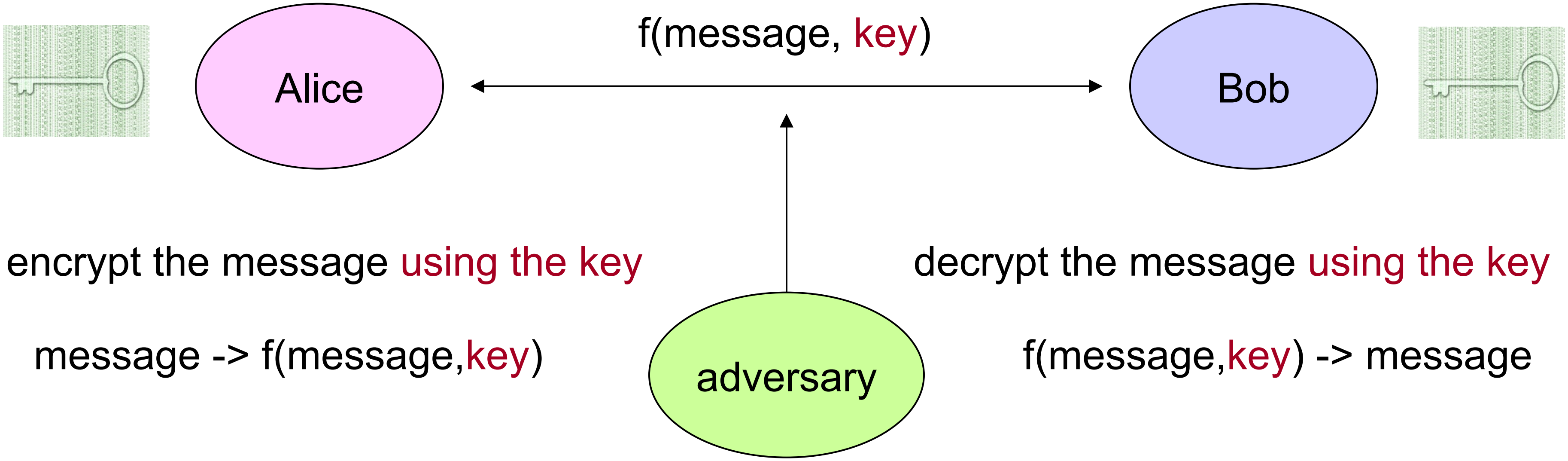
$m^*m' \equiv k \pmod{p}$

So the adversary can figure out k.

Outline

- Introduction to Cryptograph
- Turing Code
- **Public Key Cryptography**
- RSA Cryptosystem

Private Key Cryptosystem



encrypt the message using the key

message -> f(message,key)

decrypt the message using the key

f(message,key) -> message

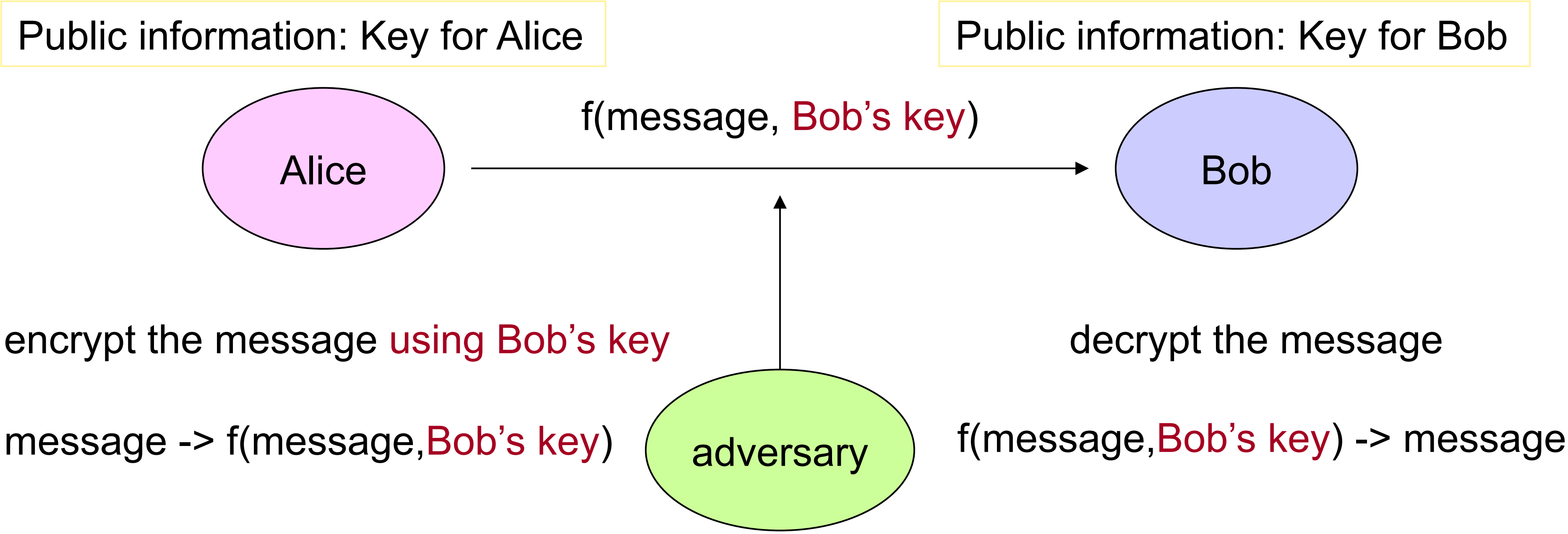
But the adversary can not decrypt f(message,key) without the key

Two parties have to agree on a **secret key**, which may be difficult in practice.

If we buy books from Amazon, we don’t need to exchange a secret code.

Why is it secure?

Private Key Cryptosystem



Public information: Key for Alice

Public information: Key for Bob

f(message, Bob's key)

encrypt the message using Bob's key

message -> f(message,Bob's key)

decrypt the message

f(message,Bob's key) -> message

But the adversary can not decrypt f(message, Bob's key)!

Only Bob can decrypt the message sent to him!

There is no need to have a secret key between Alice and Bob.

How is it possible???

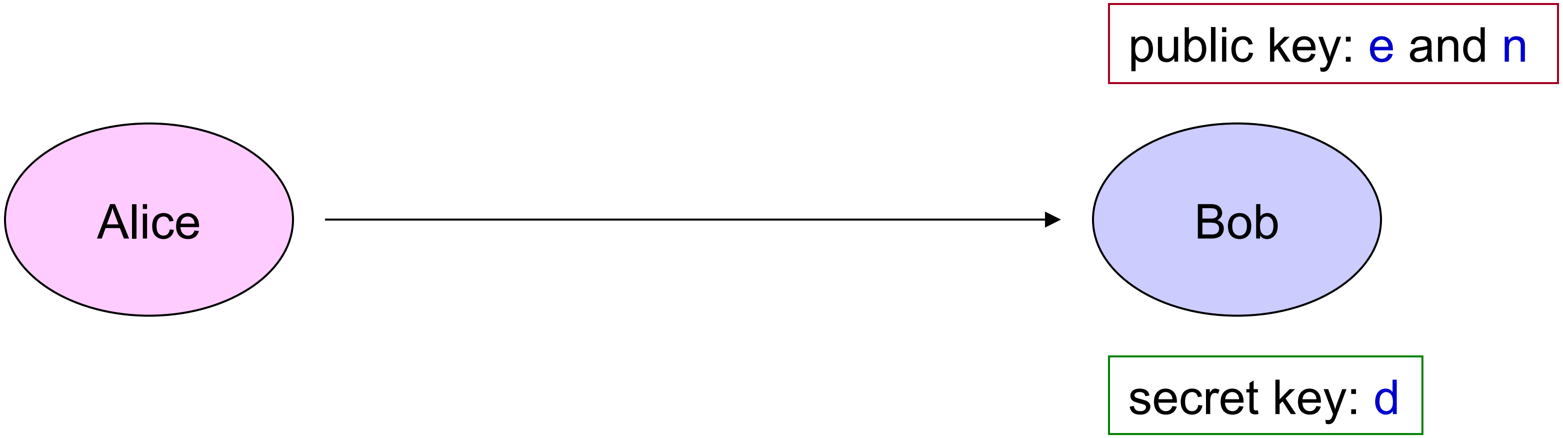
Outline

- Introduction to Cryptograph
- Turing Code
- Public Key Cryptography
- **RSA Cryptosystem**

RSA Cryptosystem

RSA are the initials of three Computer Scientists, Ron Rivest, Adi Shamir and Len Adleman, who discovered their algorithm when they were working together at MIT in 1977.

Generating Public Key



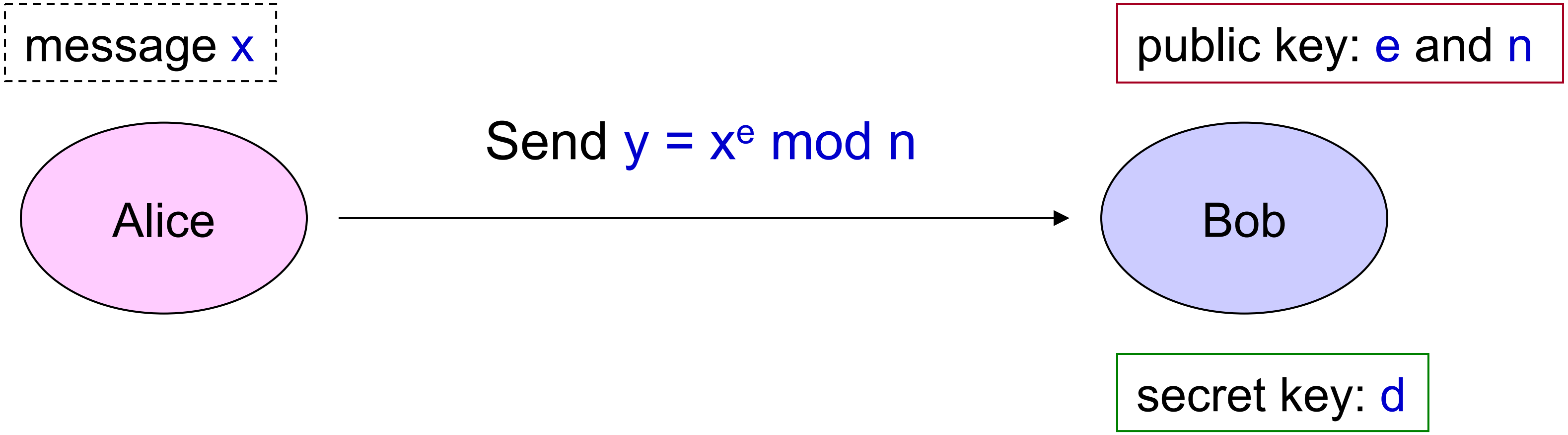
How Bob creates his public keys?

Secret key only known to Bob.

- Choose 2 large prime numbers p and q.
- Set $n = pq$ and $T = (p-1)(q-1)$
- Choose $e \neq 1$ so that $\gcd(e,T)=1$
- Calculate d so that $de \equiv 1 \pmod{T}$
- Publish e and n as public keys
- Keep d as secret key

> 150 digits

Encrypting Message

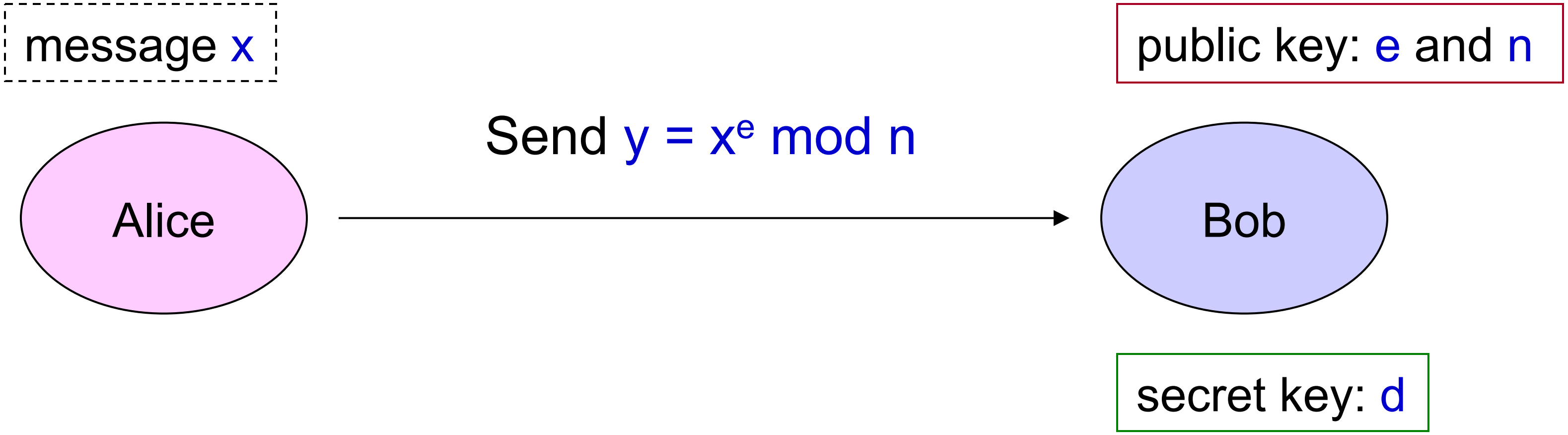


How Alice sends a message to Bob?

- Look at Bob's homepage for e and n.
- Send $y = x^e \bmod n$

Alice does not need to know Bob's secret key to send the message.

Decrypting Message



message x

public key: e and n

Send $y = x^e \bmod n$

Alice

Bob

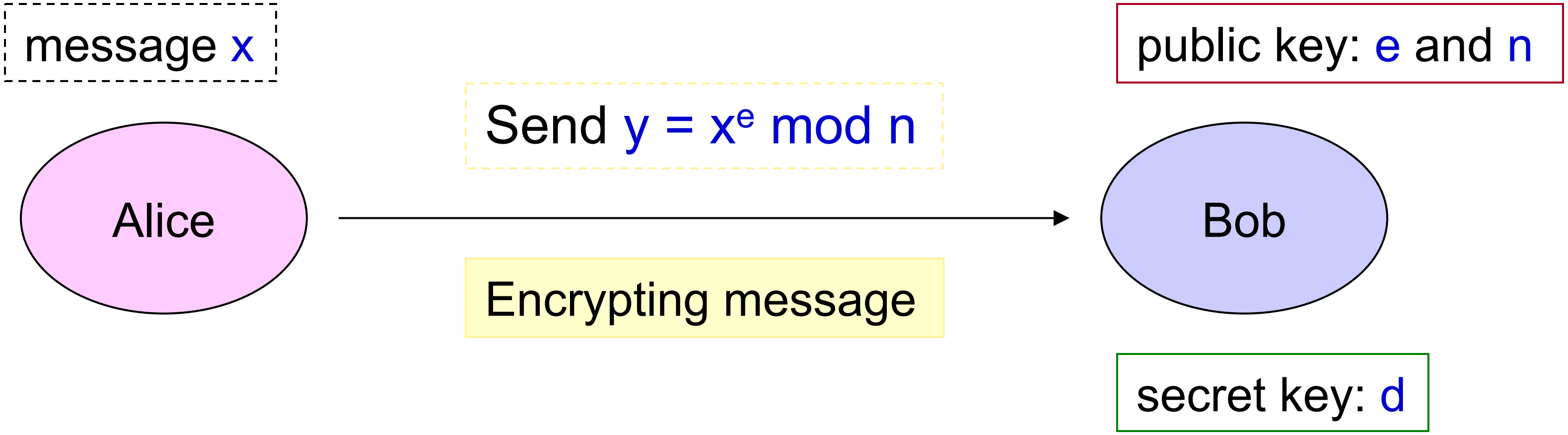
secret key: d

How Bob recover Alice's message?

- Receive $y = x^e \bmod n$
- Compute $z = y^d \bmod n$

Bob uses z is the original message that Alice sent.

RSA Cryptosystem



message x

Send $y = x^e \bmod n$

Alice

Encrypting message

Bob

public key: e and n

secret key: d

Key generation

- Choose 2 large prime numbers p and q.
- Set $n = pq$ and $T = (p-1)(q-1)$
- Choose $e \neq 1$ so that $\gcd(e,T)=1$
- Calculate d so that $de \equiv 1 \pmod{T}$
- Publish e and n as public keys
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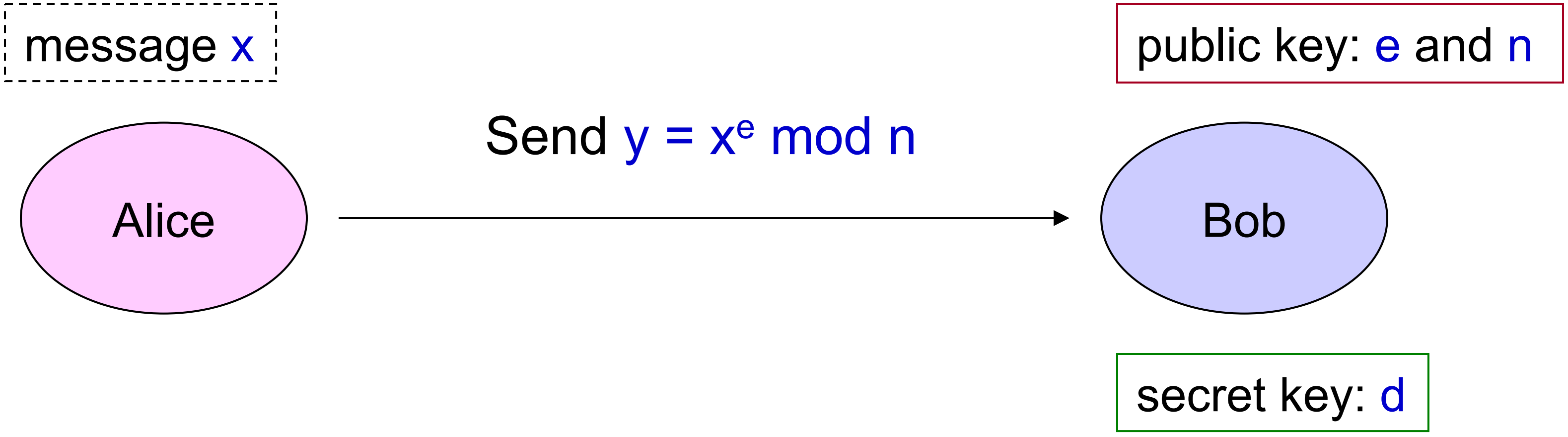
Decrypting message

Compute $z = y^d \bmod n$

RSA Cryptosystem



message x

public key: e and n

Send $y = x^e \bmod n$

secret key: d

Compute $z = y^d \bmod n$

For the RSA cryptosytem to work,

we need to show:

1) $z = x$

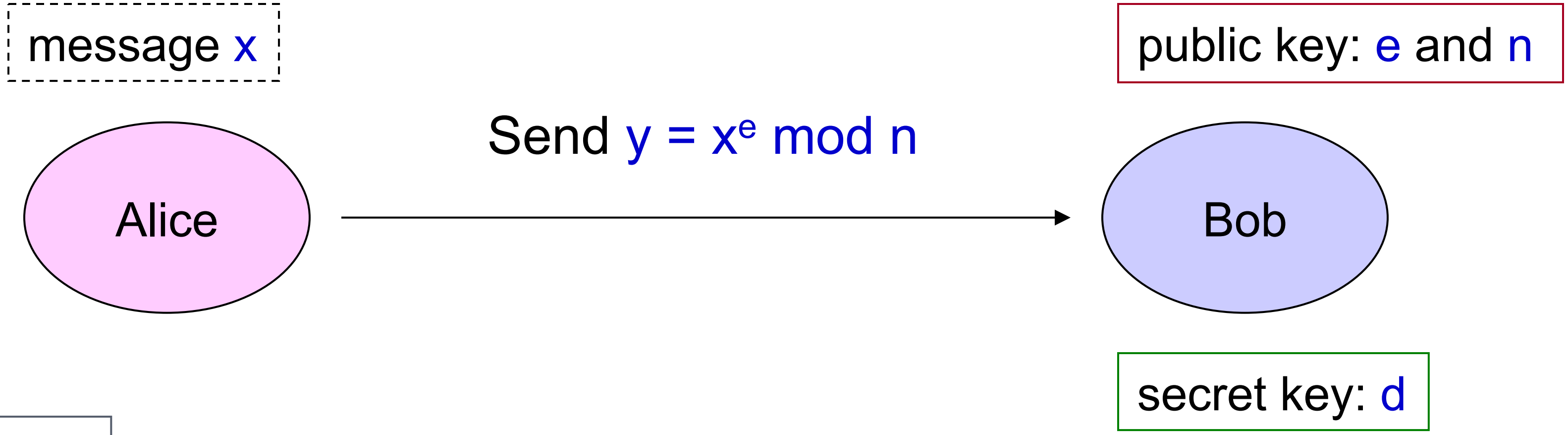
2) Without the secret key d,

we can not compute the original message

before the sun burns out.

with additional assumptions…

Correctness



message x

public key: e and n

Send $y = x^e \bmod n$

Alice

Bob

secret key: d

1) $z = x$

Compute $z = y^d \bmod n$

Note that $z = y^d \bmod n = x^{ed} \bmod n.$

Therefore we need to prove $x = x^{ed} \bmod n.$

(a) $x \bmod p = x^{ed} \bmod p$

(b) $x \bmod q = x^{ed} \bmod q$

(c) $x \bmod n = x^{ed} \bmod n$

p, q prime

$n = pq$

$T = (p-1)(q-1)$

e s.t. $\gcd(e,T)=1$

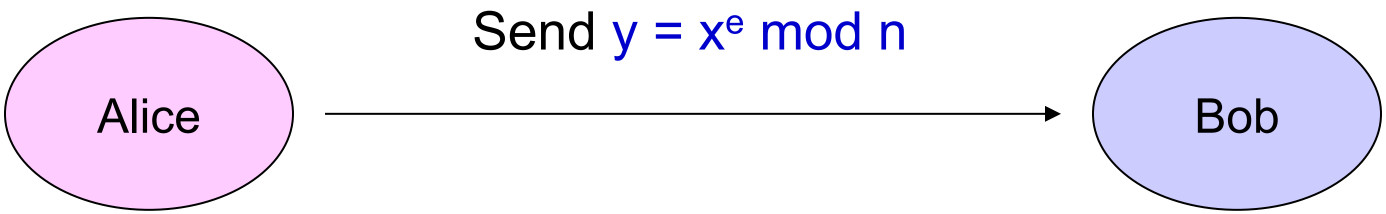
$de \equiv 1 \pmod{T}$

Therefore, if Alice sends $x < n$, then Bob can recover correctly.

Correctness

message x

public key: e and n



Send $y = x^e \bmod n$

Alice

Bob

secret key: d

1) $z = x$

(a) $x \bmod p = x^{ed} \bmod p$

Compute $z = y^d \bmod n$

Note that $de = 1 + kT \quad = 1 + k(p-1)(q-1)$

p, q prime

$n = pq$

$T = (p-1)(q-1)$

e s.t. $\gcd(e,T)=1$

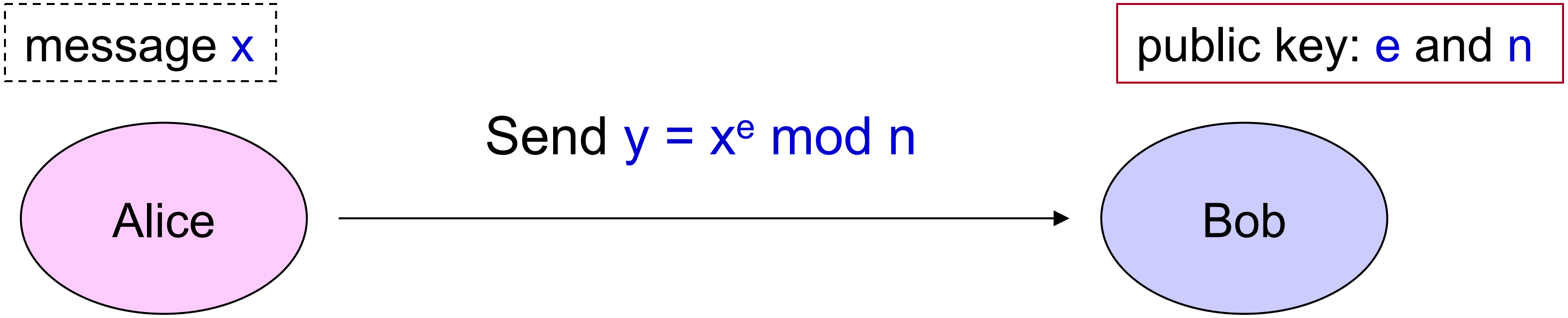
$de \equiv 1 \pmod{T}$

Hence, $x^{ed} \bmod p = x^{1+k(p-1)(q-1)} \bmod p$

$= x \cdot x^{k(p-1)(q-1)} \bmod p$

$= x \cdot (x^{k(q-1)})^{(p-1)} \bmod p$

Correctness



message x

public key: e and n

Send $y = x^e \bmod n$

Alice

Bob

secret key: d

1) $z = x$

(a) $x \bmod p = x^{ed} \bmod p$

Compute $z = y^d \bmod n$

Fermat's little theorem: If $p \nmid a$, then $a^{p-1} \equiv 1 \bmod p$

Hence, $x^{ed} \bmod p = x^{1+k(p-1)(q-1)} \bmod p$

$$= x \cdot x^{k(p-1)(q-1)} \bmod p$$

$$= x \cdot (\underbrace{x^{k(q-1)}}_{a})^{(p-1)} \bmod p$$

$$= x \bmod p$$

p, q prime

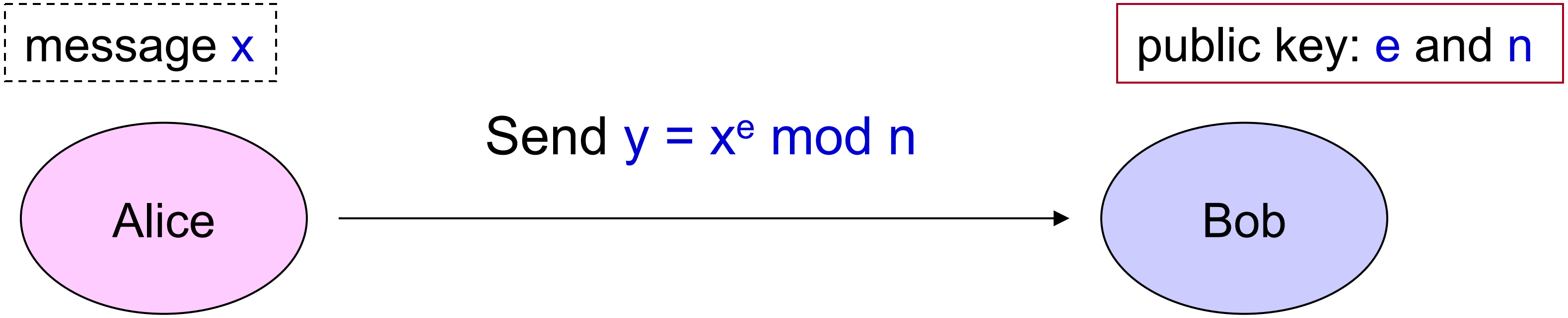
$n = pq$

$T = (p-1)(q-1)$

e s.t. $\gcd(e,T)=1$

$de \equiv 1 \pmod{T}$

Correctness



secret key: d

1) $z = x$

(a) $x \bmod p = x^{ed} \bmod p$

Compute $z = y^d \bmod n$

This means $p \mid x^{k(q-1)}$, implying $p \mid x$, since p is prime

p, q prime

$n = pq$

$T = (p-1)(q-1)$

e s.t. $\gcd(e,T)=1$

$de \equiv 1 \pmod{T}$

Hence, $x^{ed} \bmod p = x^{1+k(p-1)(q-1)} \bmod p$

$$= x \cdot x^{k(p-1)(q-1)} \bmod p$$

$$= x \cdot (\underbrace{x^{k(q-1)}}_{a})^{(p-1)} \bmod p$$

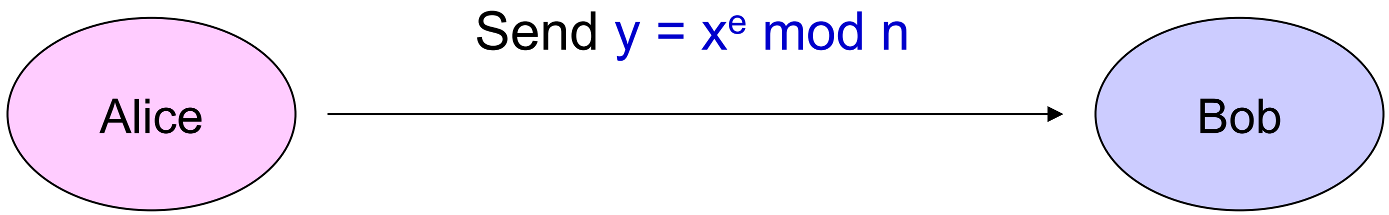
What if $p \mid a$?

Since $p \mid x$, we have $x^{ed} \bmod p = x \bmod p = 0$.

Correctness

message x

public key: e and n



Send $y = x^e \bmod n$

secret key: d

1) $z = x$

Compute $z = y^d \bmod n$

Note that $z = y^d \bmod n = x^{ed} \bmod n$.

Therefore we need to prove $x = x^{ed} \bmod n$.

(a) $x \bmod p = x^{ed} \bmod p$ ✓

(b) $x \bmod q = x^{ed} \bmod q$ ✓

The same proof.

(c) $x \bmod n = x^{ed} \bmod n$ ✓

p, q prime

$n = pq$

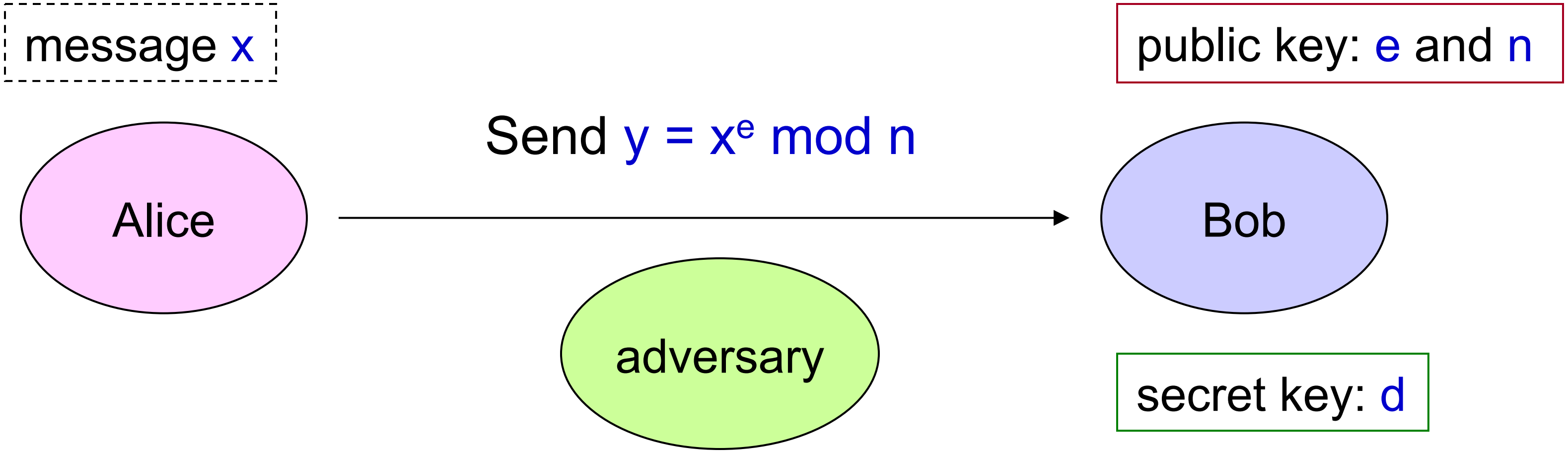
$T = (p-1)(q-1)$

e s.t. $\gcd(e,T)=1$

$de \equiv 1 \pmod{T}$

(c) can be proved directly, also follows from Chinese Remainder theorem.

Why is This Secure?



message x

public key: e and n

Send $y = x^e \bmod n$

secret key: d

Compute $z = y^d \bmod n$

2) Without the secret key d,

we can not compute the original message

before the sun burns out.

Method 1:

From $y=x^e \bmod n$, don't know how to compute x.

Thus not possible to work backward.

It is an example of an "one-way" function.

p, q prime

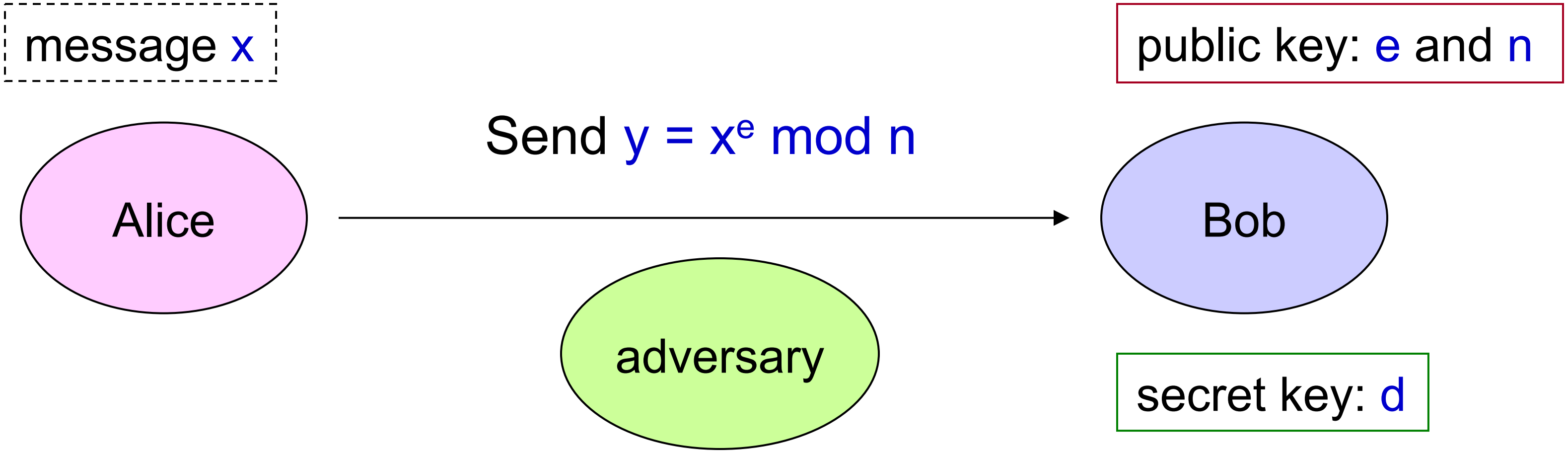
$n = pq$

$T = (p-1)(q-1)$

e s.t. $\gcd(e,T)=1$

$de \equiv 1 \pmod T$

Why is This Secure?



message x

public key: e and n

Send $y = x^e \bmod n$

Alice

Bob

adversary

secret key: d

2) Without the secret key d,

we can not compute the original message

before the sun burns out.

Compute $z = y^d \bmod n$

p, q prime

$n = pq$

$T = (p-1)(q-1)$

e s.t. $\gcd(e,T)=1$

$de \equiv 1 \pmod{T}$

Method 2:

Factor $n = pq$. Compute secrete key d.

Then decrypt everything!

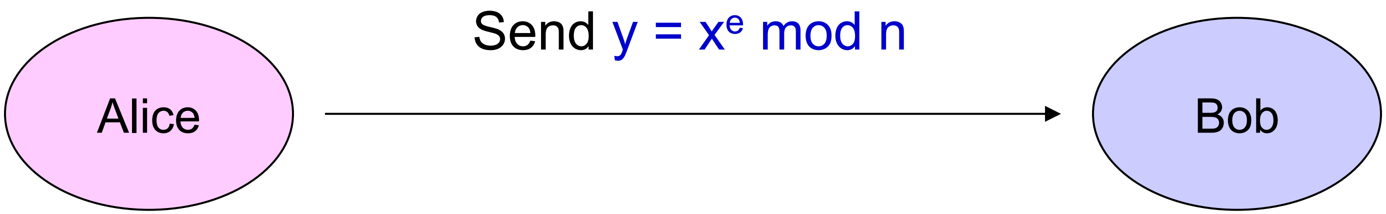
No one knows an efficient way to do factoring.

The security is based on assumptions that some computational problems are hard.

RSA Example

message x

public key: e and n


Send $y = x^e \bmod n$

secret key: d

Then Alice sends the encrypted message.

Compute $z = y^d \bmod n$

$x=33$ $y = 33^{23} \bmod 55$

$y = 84298649517881922539738734663399137 \bmod 55$

How to compute it efficiently?

$p=5$ $q=11$

$n = 55$

$T = 40$

$e = 7$

$d = 23$

→

p, q prime

$n = pq$

$T = (p-1)(q-1)$

e s.t. $\gcd(e,T)=1$

$de \equiv 1 \pmod{T}$

First Bob generated his keys.

Exponentiation

To compute exponentiation mod n

144^4 mod 713

= 144 * 144 * 144 * 144 mod 713

= 20736 * 144 * 144 mod 713

= 59 * 144 * 144 mod 713

= 8496 * 144 mod 713

= 653 * 144 mod 713

= 94032 mod 713

= 629 mod 713

This still takes too long when the exponent is large.

20736 * 20736 mod 713

= 59 * 59 mod 713

= 3481 mod 713

= 629 mod 713

This is much more efficient.

Repeated Squaring

Note that 50 = 32 + 16 + 2

144^{50} mod 713

= $144^{32}\ 144^{16}\ 144^{2}$ mod 713

= $648 \cdot 485 \cdot 59$ mod 713

= 242

144^2 mod 713 = 59

144^4 mod 713
= $144^2 \cdot 144^2$ mod 713
= $59 \cdot 59$ mod 713
= 629

144^8 mod 713
= $144^4 \cdot 144^4$ mod 713
= $629 \cdot 629$ mod 713
= 639

144^{16} mod 713
= $144^8 \cdot 144^8$ mod 713
= $639 \cdot 639$ mod 713
= 485

144^{32} mod 713
= $144^{16} \cdot 144^{16}$ mod 713
= $485 \cdot 485$ mod 713
= 648

Remarks

- We have derived everything from basic principle.

- RSA cryptosystem is one of the most important achievements in compute science.
 (The researchers won the Turing award for their contribution.)

- Number theory is also very useful in coding theory (e.g. compression).

- Mathematics is very important in computer science.

Remarks

Theorem: if n is composite, for more than half of $a < n$,

the strong primality test will say n is composite!

The proof uses Chinese Remainder theorem and some elementary number theory. (Introduction to Algorithms, MIT press)

Conjecture: It is enough to try a to up to roughly $\log(n)$.

Theroem (Primes is in P, 2004)

There is an efficient and deterministic primality test.

Major Open Problem:

Is there an efficient algorithm to compute the prime factorization?

Next class

- Topic: Proof by Mathematical Induction
- Pre-class reading: Chap 5.1