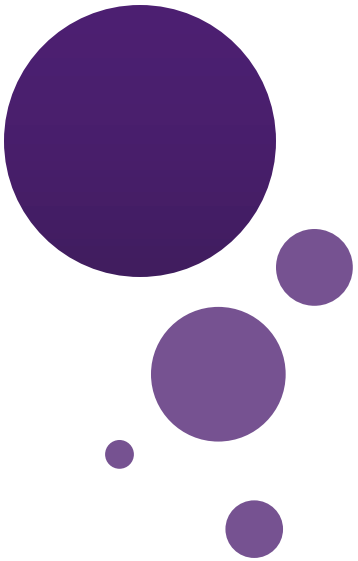
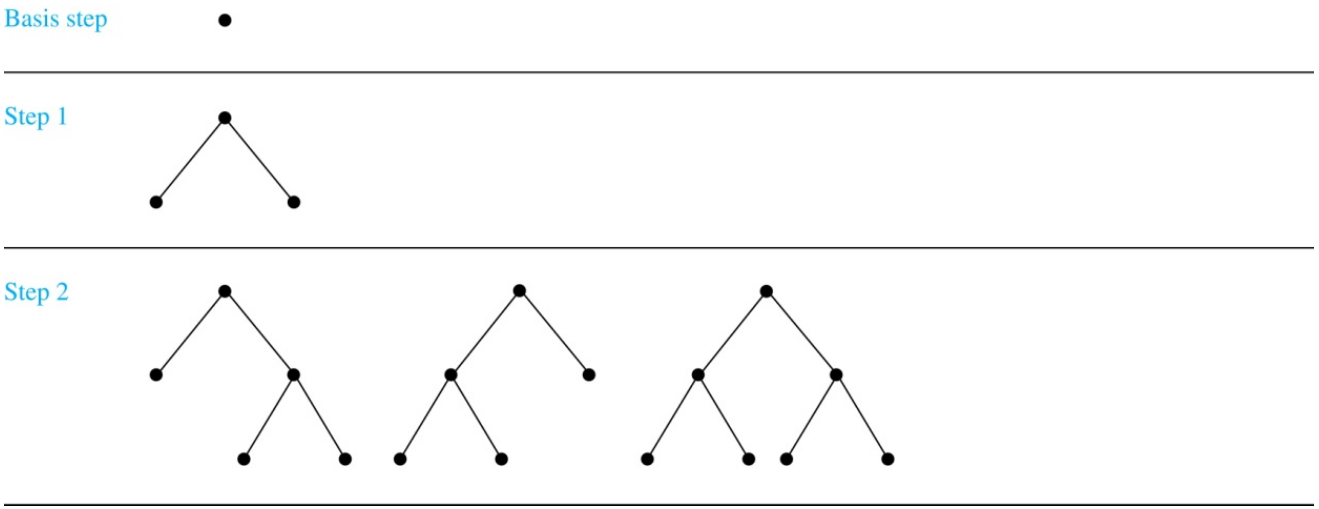


Lecture 24: Recursive Algorithms

Dr. Chengjiang Long
Computer Vision Researcher at Kitware Inc.
Adjunct Professor at SUNY at Albany.
Email: **clong2@albany.edu**

Recap Previous Lecture

- Recursion Definition
  - Basis Step: Specify the value of the function for the base case.
  - Recursive Step: Give a rule for finding the value of a function from its values at smaller integers greater than the base case.
- Structure Induction



Outline

- Definition of Recursive Algorithms
- Several Recursive Algorithms
- Efficiency of Recursive Algorithms

Outline

- **Definition of Recursive Algorithms**
- Several Recursive Algorithms
- Efficiency of Recursive Algorithms

Recursive Algorithms

- Recursive definitions can be used to describe functions and sets as well as algorithms.
- A recursive procedure is a procedure that invokes itself.
- A recursive algorithm is an algorithm that contains a recursive procedure.
- An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

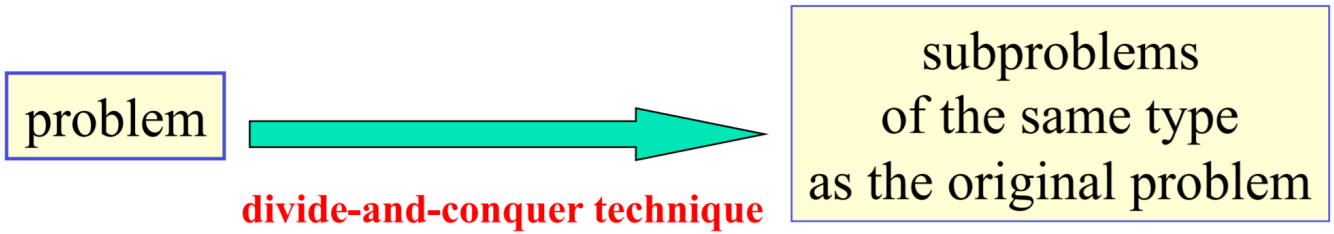
Example

- A procedure to compute a^n.

  procedure *power*(a≠0: real, n∈N)

  if n = 0 **then return** 1

  else return a·*power*(a, n−1)



- ❑Note recursive algorithms are often simpler to code than iterative ones…
- ❑However, they can consume more stack space if your compiler is not smart enough

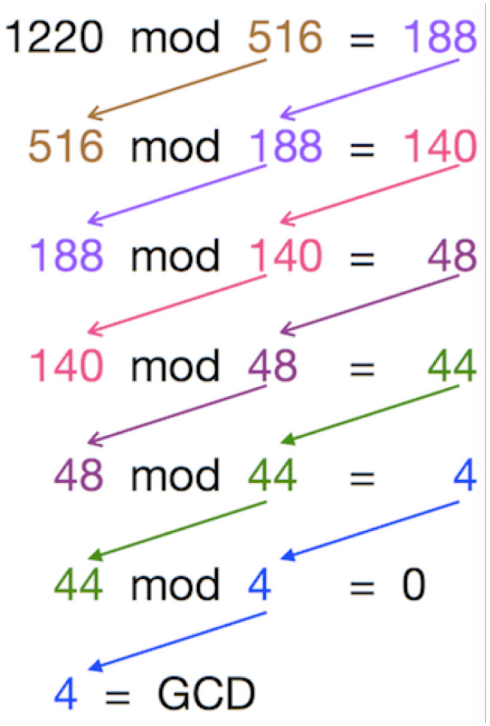
Outline

- Definition of Recursive Algorithms
- **Several Recursive Algorithms**
- Efficiency of Recursive Algorithms

Recursive Euclid's Algorithm

- gcd(a, b) = gcd((b mod a), a)

procedure gcd(a,b∈N with a < b)
if a = 0 **then return** b
else return gcd(b mod a, a)



Recursive Linear Search

- {Finds *x* in series *a* at a location ≥*i* and ≤*j* }

  procedure *search*

  (*a*: series; *i*, *j*: integer; *x*: item to find)

  if a_i = *x* **return** *i* {At the right item? Return it!}

  if *i* = *j* **return** 0 {No locations in range? Failure!}

  return *search*(*a*, *i* +1, *j*, *x*) {Try rest of range}

- Note there is no real advantage to using recursion here over just looping

  for *loc* := *i* to *j*…

- Recursion is slower because procedure call costs

Recursive Binary Search

{Find location of x in a, $\geq i$ and $\leq j$}

procedure *binarySearch*(a, x, i, j)

$m := \lfloor (i + j)/2 \rfloor$ {Go to halfway point}

if $x = am$ **return** m {Did we luck out?}

if $x < am \wedge i < m$ {If it's to the left, check that .}

return *binarySearch*(a, x, i, $m-1$)

else if $x > am \wedge j > m$ {If it's to right, check that .}

return *binarySearch*(a, x, $m+1$, j)

else return 0 {No more items, failure.}

Recursive Fibonacci Algorithm

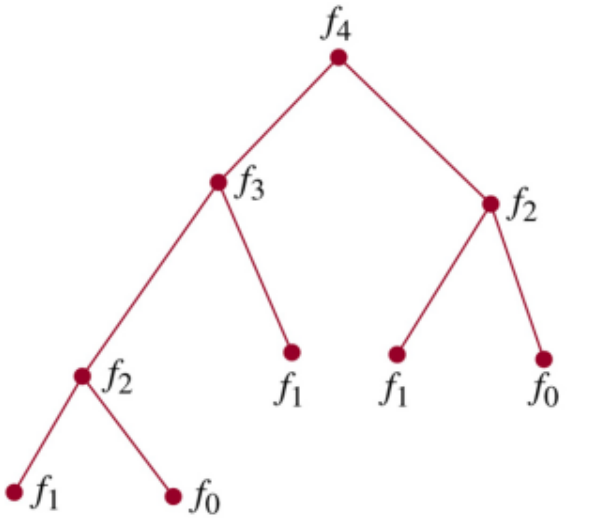
procedure *fibonacci*($n \in$ **N**)

if $n = 0$ **return** 0

if $n = 1$ **return** 1

return *fibonacci*($n - 1$) + *fibonacci*($n - 2$)

- Is this an efficient algorithm?
- How many additions are performed?



Analysis of Fibonacci Procedure

Theorem: The recursive procedure *fibonacci*(*n*) performs $f_n+1 - 1$ additions.

- **Proof:** By strong structural induction over *n*, based on the procedure's own recursive definition.
- **Basis step:**
  - *fibonacci*(0) performs 0 additions, and $f_0+1 - 1 = f_1 - 1 = 1 - 1 = 0$.
  - Likewise, *fibonacci*(1) performs 0 additions, and $f_1+1 - 1 = f_2 - 1 = 1 - 1 = 0$.

Analysis of Fibonacci Procedure

Inductive step:

$$fibonacci(k+1) = \underbrace{fibonacci(k)}_{\text{by } P(k):\ f_{k+1} - 1 \text{ additions}} + \underbrace{fibonacci(k-1)}_{\text{by } P(k-1):\ f_k - 1 \text{ additions}}$$

- For $k > 1$, by strong inductive hypothesis, *fibonacci*(k) and *fibonacci*($k-1$) do $f_{k+1} - 1$ and $f_k - 1$ additions respectively.
- *fibonacci*($k+1$) adds 1 more, for a total of

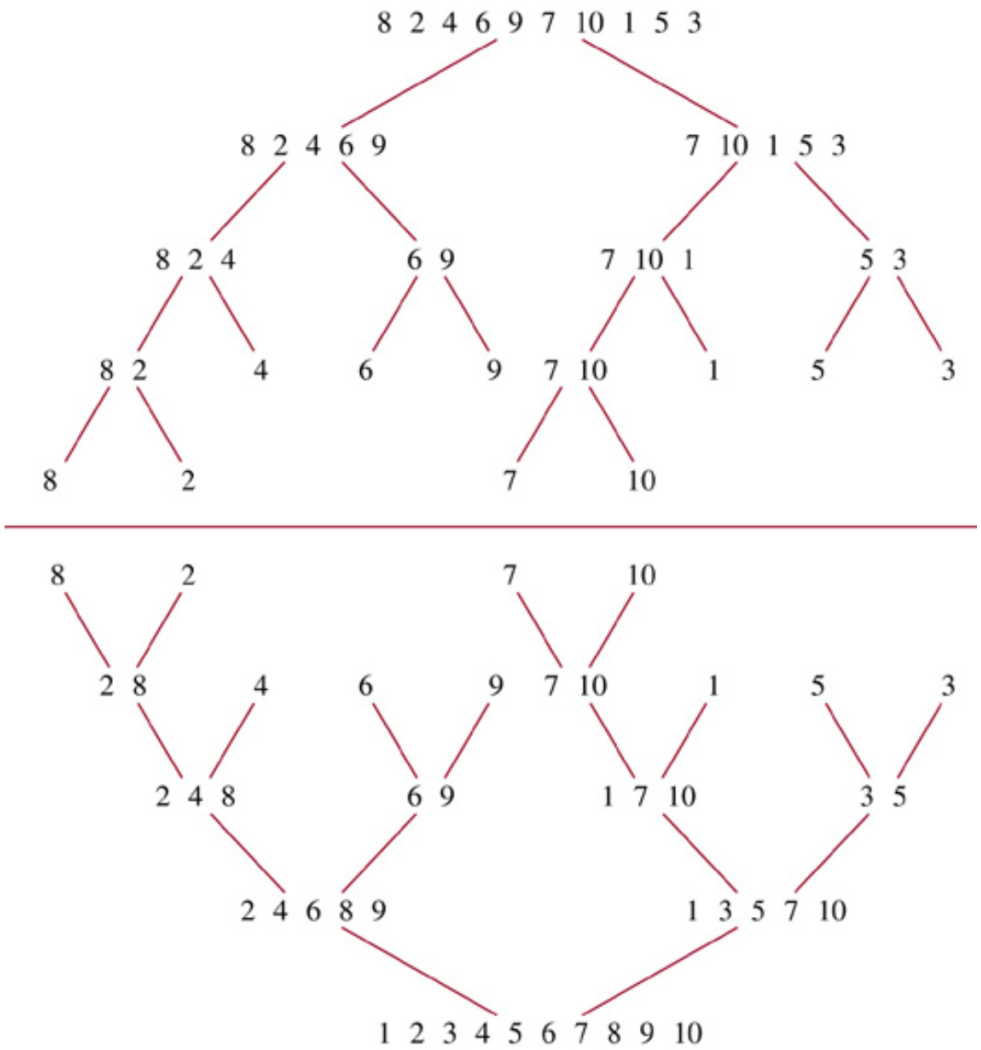
$$(f_{k+1} - 1) + (f_k - 1) + 1 = f_{k+1} + f_k - 1$$
$$= f_{k+2} - 1.$$

Iterative Fibonacci Algorithm

- **procedure** *iterativeFib*($n \in$ **N**)
- **if** $n = 0$ **then**
- **return** 0
- **else begin**
- $x := 0$
- $y := 1$
- **for** $i := 1$ **to** $n - 1$ **begin**
- $z := x + y$
- $x := y$
- $y := z$
- **end**
- **end**
- **return** y {the *n*th Fibonacci number}

Requires only $n - 1$ additions

Recursive Merge Sort Example



Split

Merge

Recursive Merge Sort

procedure *mergesort*(L = 1,…, n)

if $n > 1$ **then**

$m := \lfloor n/2 \rfloor$ {this is rough ½-way point}

$L1 := 1,\ldots, m$

$L2 := m+1,\ldots, n$

$L := merge(mergesort(L1), mergesort(L2))$

return L

- The merge takes $\Theta(n)$ steps, and therefore the merge-sort takes $\Theta(n \log n)$.

Merging Two Sorted Lists

TABLE 1 **Merging the Two Sorted Lists 2, 3, 5, 6 and 1, 4.**

First List	*Second List*	*Merged List*	*Comparison*
2 3 5 6	1 4		1 < 2
2 3 5 6	4	1	2 < 4
3 5 6	4	1 2	3 < 4
5 6	4	1 2 3	4 < 5
5 6		1 2 3 4	
		1 2 3 4 5 6	

Recursive Merge Method

{Given two sorted lists $A = (a_1, \ldots, a_{|A|})$, $B = (b_1, \ldots, b_{|B|})$, returns a sorted list of all.}

procedure *merge*(A, B: sorted lists)

if A = empty **return** B {If A is empty, it's B.}

if B = empty **return** A {If B is empty, it's A.}

if $a_1 < b_1$ **then**

return $(a_1, merge((a_2, \ldots, a_{|A|}), B))$

else

return $(b_1, merge(A, (b_2, \ldots, b_{|B|})))$

Outline

- Definition of Recursive Algorithms
- Several Recursive Algorithms
- **Efficiency of Recursive Algorithms**

Efficiency of Recursive Algorithms

- The time complexity of a recursive algorithm may depend critically on the number of recursive calls it makes.
- **Example:** *Modular exponentiation* to a power *n* can take log(*n*) time if done right, but linear time if done slightly differently.
- Task: Compute b^n **mod** *m*, where *m*≥2, *n*≥0, and 1≤*b*<*m*.

Modular Exponentiation #1

- Uses the fact that $b^n = b \cdot b^{n-1}$ and that
  $x \cdot y$ **mod** $m = x \cdot (y$ **mod** $m)$ **mod** m.
  (Prove the latter theorem at home.)
  {Returns b^n mod m.}
  procedure *mpower*
  (b, n, m: integers with $m \geq 2$, $n \geq 0$, and $1 \leq b < m$)
  if $n=0$ **then return** 1 **else**
  return $(b \cdot mpower(b, n-1, m))$ **mod** m

- Note this algorithm takes $\Theta(n)$ steps!

Modular Exponentiation #2

- Uses the fact that $b^{2k} = b^{k \cdot 2} = (b^k)^2$
- Then, b^{2k} **mod** $m = (b^k$ **mod** $m)^2$**mod** m.

procedure *mpower*(*b*,*n*,*m*) *{same signature}*
if *n*=0 **then return** 1
else if 2|*n* **then**
return *mpower*(*b*,*n*/2,*m*)2 **mod** *m*
else return (*b*·*mpower*(*b*,*n*−1,*m*)) **mod** *m*

- What is its time complexity? **Θ(log *n*) steps**

A Slight Variation

- Nearly identical but takes $\Theta(n)$ time instead!

procedure *mpower*(*b*,*n*,*m*) {same signature}
if $n=0$ **then return** 1
else if $2|n$ **then**
return (*mpower*(*b*,*n*/2,*m*)·
mpower(*b*,*n*/2,*m*)) **mod** *m*
else return (*mpower*(*b*,*n*−1,*m*)·*b*) **mod** *m*

The number of recursive calls made is critical!

Next class

- Topic: Recursive Algorithms and Basic Counting Rules
- Pre-class reading: Chap 5.4 and Chap 6.1