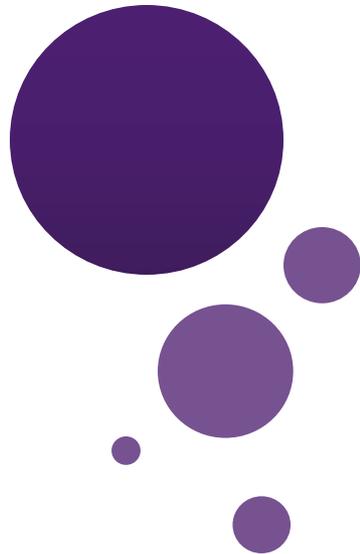




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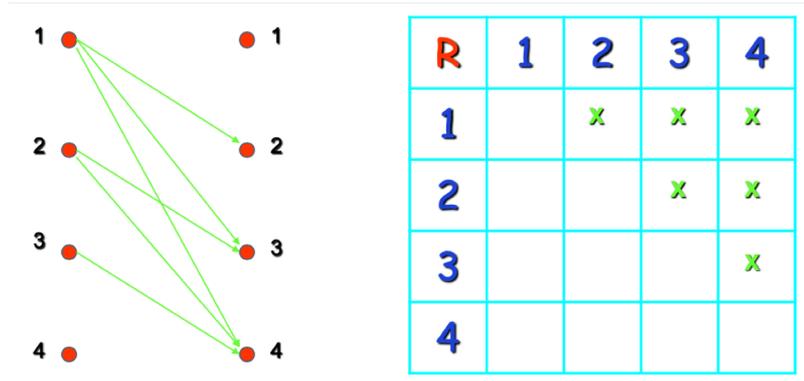
## Lecture 34: Graph Theory (1)



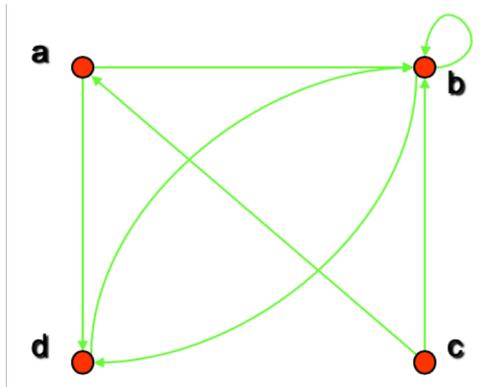
Dr. Chengjiang Long  
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# Recap Previous Lecture

- Relations and its Representation
- Equivalence Relations and Equivalence Classes



$V = \{a, b, c, d\}$ ,  
 $E = \{(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)\}$ .



Let  $R$  be the relation  $\{(a, b) \mid a \equiv b \pmod{3}\}$  on the set of integers. Is  $R$  an equivalence relation?

•Yes,  $R$  is reflexive, symmetric, and transitive.

•What are the equivalence classes of  $R$  ?

• $\{\dots, -6, -3, 0, 3, 6, \dots\}$ ,  
 $\{\dots, -5, -2, 1, 4, 7, \dots\}$ ,  
 $\{\dots, -4, -1, 2, 5, 8, \dots\}$

# Outline

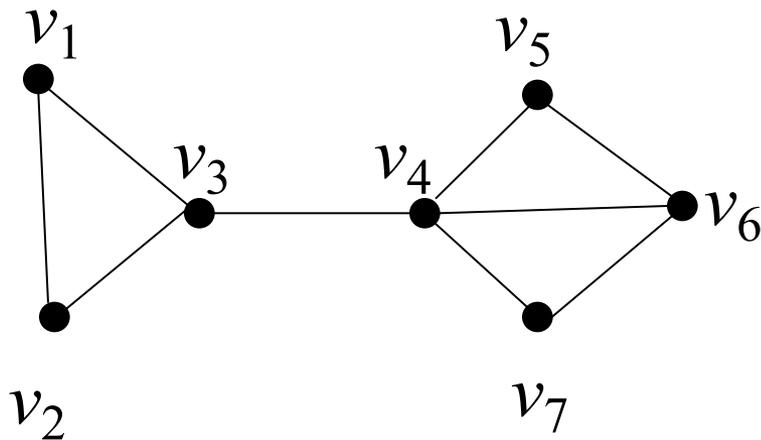
- Graph and Graph Models
- Graph Terminology and Special Types of Graphs
- Representing Graphs and Graph Isomorphism

# Outline

- **Graph and Graph Models**
- Graph Terminology and Special Types of Graphs
- Representing Graphs and Graph Isomorphism

# Introduction to Graphs

**Def 1.** A graph  $G = (V, E)$  consists of  $V$ , a nonempty set of vertices (or nodes), and  $E$ , a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.



$G = (V, E)$ , where

$V = \{v_1, v_2, \dots, v_7\}$

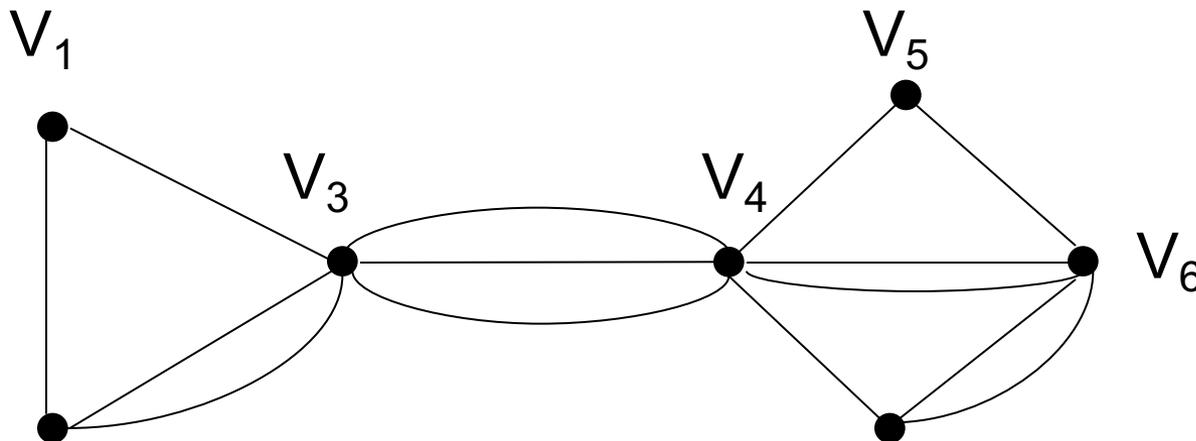
$E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\},$   
 $\{v_3, v_4\}, \{v_4, v_5\}, \{v_4, v_6\}$   
 $\{v_4, v_7\}, \{v_5, v_6\}, \{v_6, v_7\}\}$

# Introduction to Graphs

- A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a **simple graph**.

## Multigraph:

simple graph + multiple edges (**multiedges**)  
(Between two points to allow multiple edges)

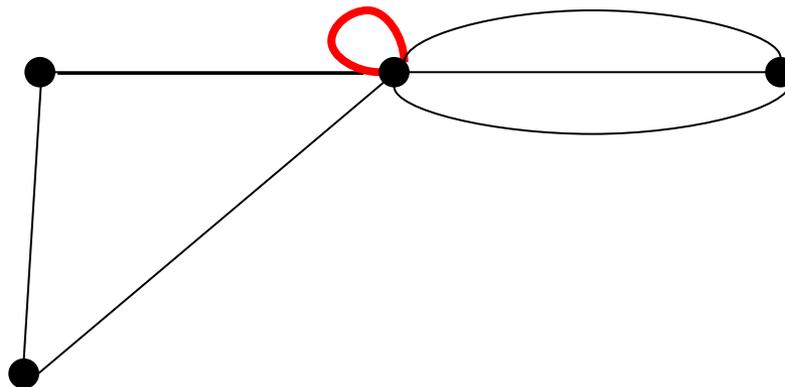


# Introduction to Graphs

**Def.** Pseudograph:

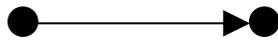
simple graph + multiedge  
+ loop

(a loop: )

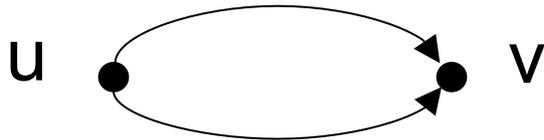


# Introduction to Graphs

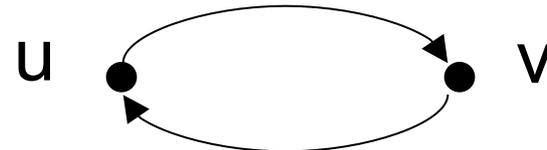
**Def 2.** Directed graph (digraph):  
simple graph with each edge directed



Note:  is allowed in a directed graph



The two edges  $(u,v), (u,v)$   
are multiedges.



The two edges  $(u,v), (v,u)$   
are not multiedges.

**Def.** Directed multigraph: digraph+multiedges

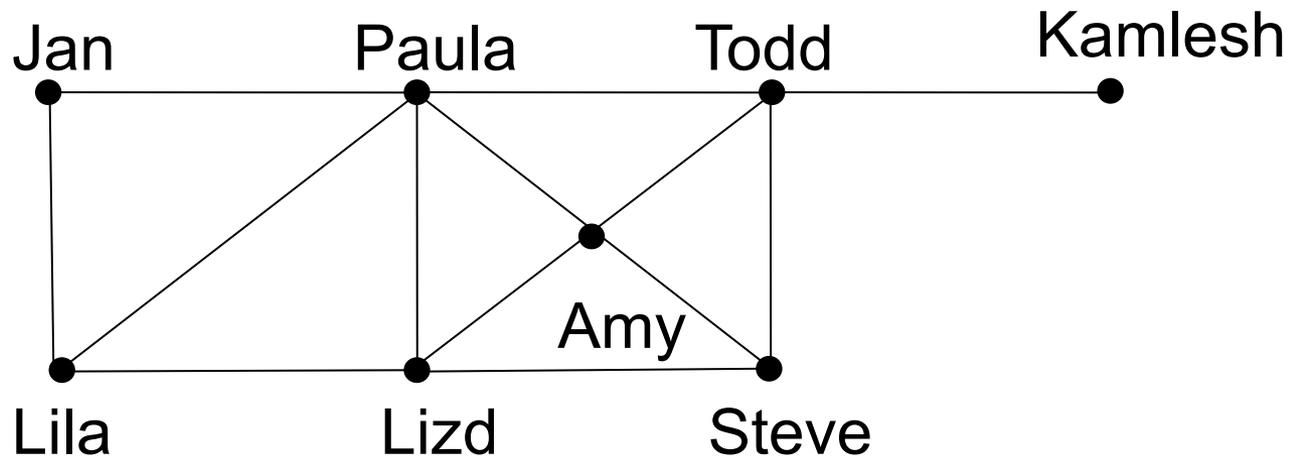
# Introduction to Graphs

**Table 1.** Graph Terminology

Type	Edges	Multiple Edges	Loops
(simple) graph	undirected edge: $\{u,v\}$	x	x
Multigraph		✓	x
Pseudograph		✓	✓
Directed graph	directed edge: $(u,v)$	x	✓
Directed multigraph		✓	✓

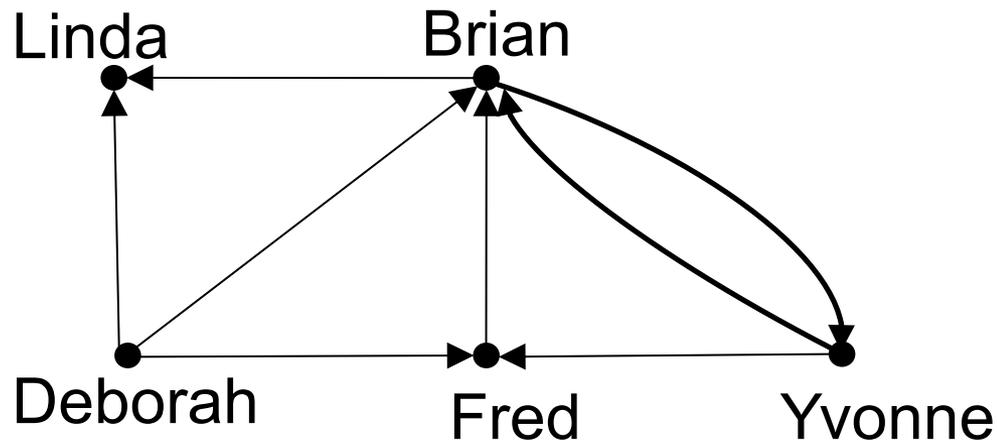
# Graph Models: Acquaintance graph

- We can use a simple graph to represent whether two people know each other. Each person is represented by a vertex. An undirected edge is used to connect two people when these people know each other.



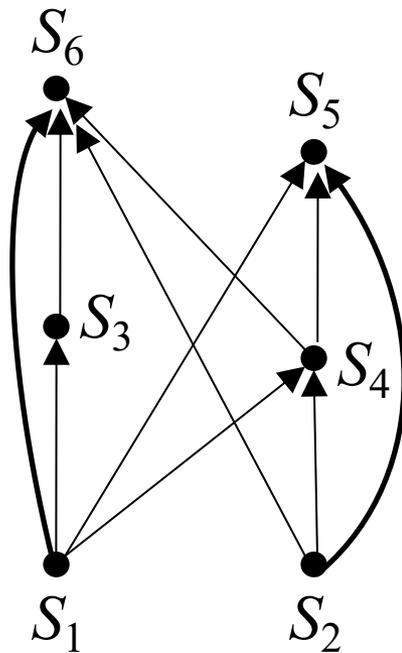
# Graph Models: Influence graph

- In studies of group behavior it is observed that certain people can influence the thinking of others.  
Simple digraph  $\Rightarrow$  Each person of the group is represented by a vertex. There is a directed edge from vertex  $a$  to vertex  $b$  when the person  $a$  influences the person  $b$ .



# Graph Models: Precedence graph and concurrent processing

- Computer programs can be executed more rapidly by executing certain statements concurrently. It is important not to execute a statement that requires results of statements not yet executed.
- Simple digraph  $\Rightarrow$  Each statement is represented by a vertex, and there is an edge from  $a$  to  $b$  if the statement of  $b$  cannot be executed before the statement of  $a$ .



$S_1: a:=0$

$S_2: b:=1$

$S_3: c:=a+1$

$S_4: d:=b+a$

$S_5: e:=d+1$

$S_6: e:=c+d$

# Outline

- Graph and Graph Models
- **Graph Terminology and Special Types of Graphs**
- Representing Graphs and Graph Isomorphism

# Graph Terminology

**Def 1.** Two vertices  $u$  and  $v$  in a undirected graph  $G$  are called **adjacent** (or **neighbors**) in  $G$  if  $\{u, v\}$  is an edge of  $G$ .

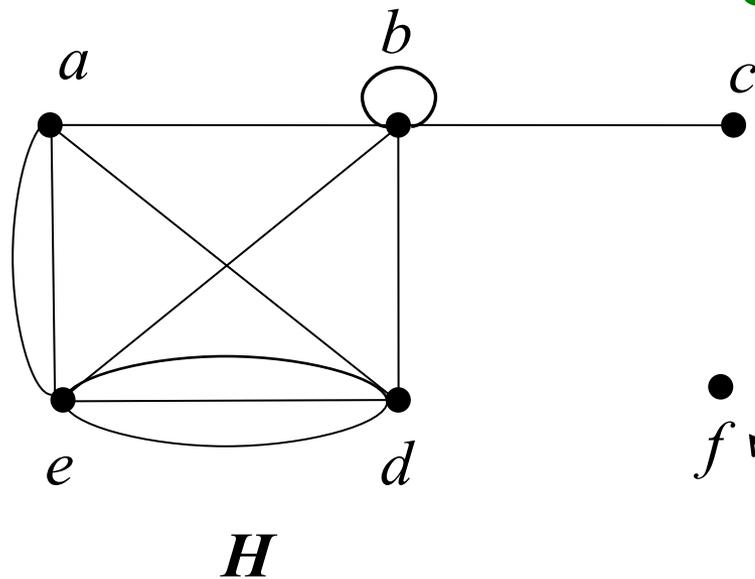
Note : **adjacent**: a vertex connected to a vertex  
**incident**: a vertex connected to an edge

**Def 2.** The **degree** of a vertex  $v$ , denoted by  $\text{deg}(v)$ , in an undirected graph is the number of edges incident with it.

(Note : A loop adds 2 to the degree.)

# Example

- What are the degrees of the vertices in the graph  $H$  ?



**Solution :**

$$\deg(a)=4$$

$$\deg(b)=6$$

$$\deg(c)=1$$

$$\deg(d)=5$$

$$\deg(e)=6$$

$$\deg(f)=0$$

**Def.** A vertex of degree 0 is called **isolated**.

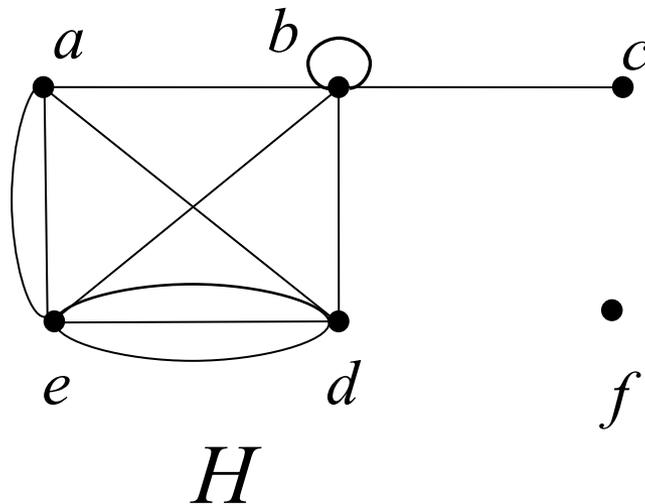
**Def.** A vertex is **pendant if and only** if it has degree one.

# The Handshaking Theorem

- Let  $G = (V, E)$  be an undirected graph with  $e$  edges (i.e.,  $|E| = e$ ). Then

$$\sum_{v \in V} \deg(v) = 2e$$

**Example:** The graph  $H$  has 11 edges, and



$$\sum_{v \in V} \deg(v) = 22$$

# Even number of odd degree

- An undirected graph  $G = (V, E)$  has an even number of vertices of odd degree.

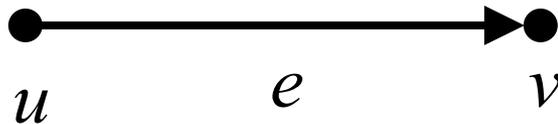
**Proof** : Let  $V_1 = \{v \in V \mid \deg(v) \text{ is even}\}$ ,  
 $V_2 = \{v \in V \mid \deg(v) \text{ is odd}\}$ .

$$2e = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

$$\Rightarrow \sum_{v \in V_2} \deg(v) \text{ is even.}$$

# Directed Graph

- Def 3.**  $G = (V, E)$ : directed graph,  
 $e = (u, v) \in E$  :  $u$  is adjacent to  $v$   
 $v$  is adjacent from  $u$
- $u$  : initial vertex of  $e$   
 $v$  : terminal (end) vertex of  $e$



The initial vertex and terminal vertex of a loop are the same



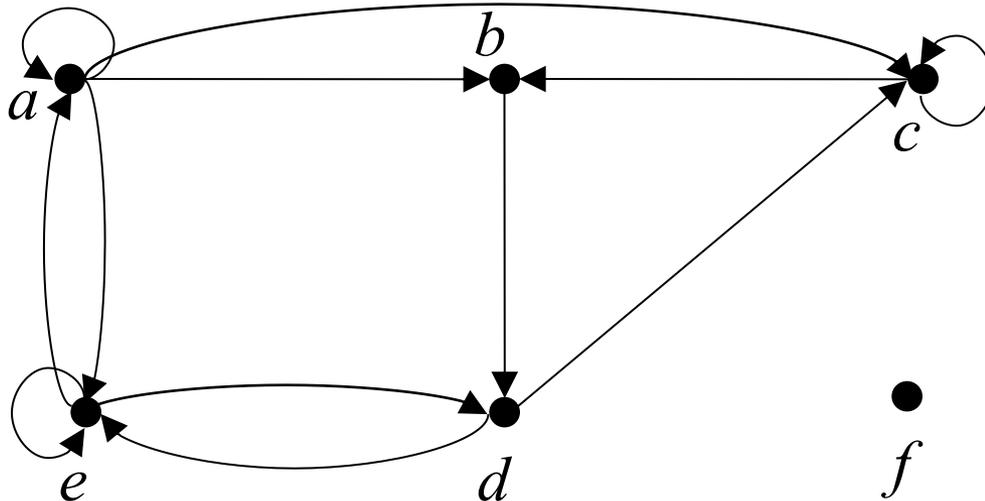
# Directed Graph

$G = (V, E)$  : directed graph,  $v \in V$

$\deg^-(v)$  : # of edges with  $v$  as a terminal. (in-degree)

$\deg^+(v)$  : # of edges with  $v$  as a initial vertex. (out-degree)

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$



$\deg^-(a)=2, \deg^+(a)=4$

$\deg^-(b)=2, \deg^+(b)=1$

$\deg^-(c)=3, \deg^+(c)=2$

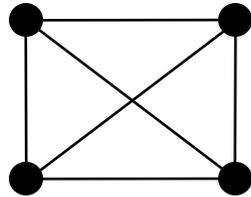
$\deg^-(d)=2, \deg^+(d)=2$

$\deg^-(e)=3, \deg^+(e)=3$

$\deg^-(f)=0, \deg^+(f)=0$

# Regular Graph

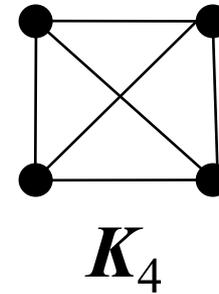
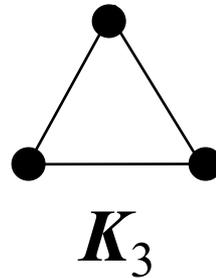
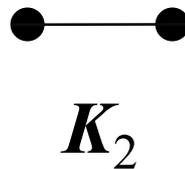
- A simple graph  $G=(V, E)$  is called **regular** if every vertex of this graph has the same degree. A regular graph is called  **$n$ -regular** if  $\deg(v)=n$  ,  $\forall v \in V$ .



is 3-regular.

# Some Special Simple Graphs

- The **complete graph on  $n$  vertices**, denoted by  $K_n$ , is the simple graph that contains exactly one edge between each pair of distinct vertices.

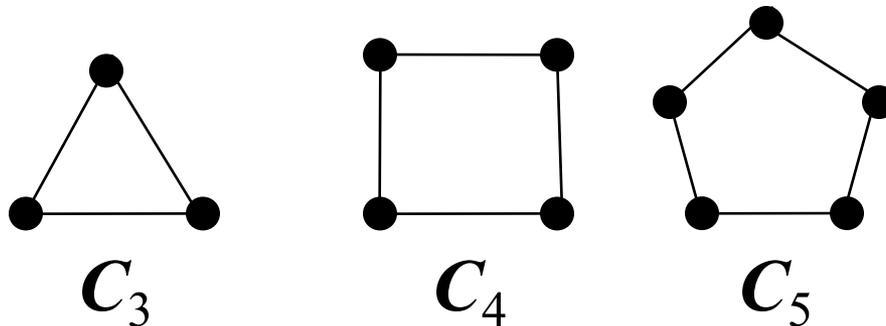


**Note.**  $K_n$  is  $(n-1)$ -regular,  $|V(K_n)|=n$ ,

$$|E(K_n)| = \binom{n}{2}$$

# Some Special Simple Graphs

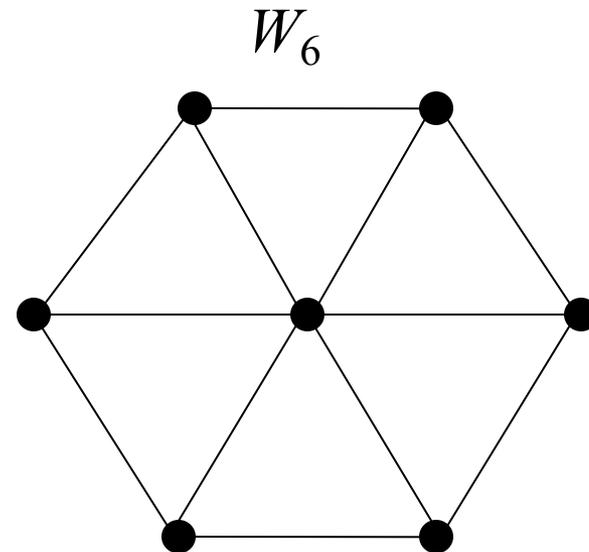
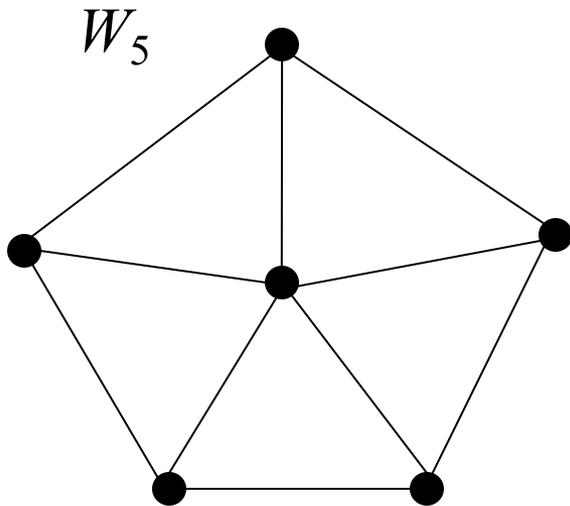
- The **cycle**  $C_n$ ,  $n \geq 3$ , consists of  $n$  vertices  $v_1, v_2, \dots, v_n$  and edges  $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$ .



**Note**  $C_n$  is 2-regular,  $|V(C_n)| = n$ ,  $|E(C_n)| = n$

# Some Special Simple Graphs

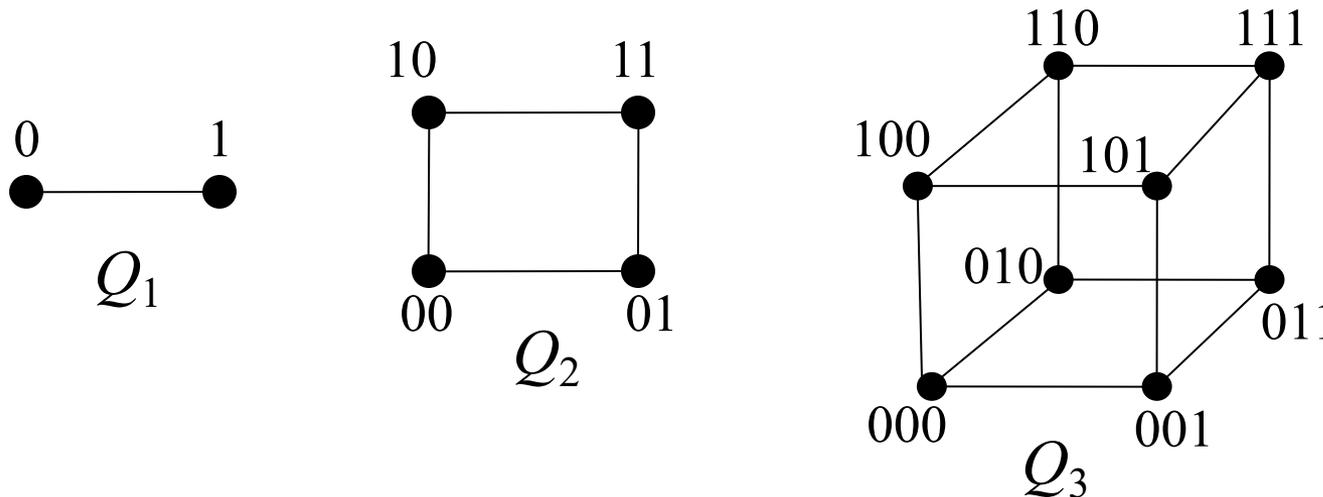
- We obtained the **wheel**  $W_n$  when we add an additional vertex to the cycle  $C_n$ , for  $n \geq 3$ , and connect this new vertex to each of the  $n$  vertices in  $C_n$ , by new edges.



**Note.**  $|V(W_n)| = n + 1$ ,  $|E(W_n)| = 2n$ ,  
 $W_n$  is not a regular graph if  $n \neq 3$ .

# Some Special Simple Graphs

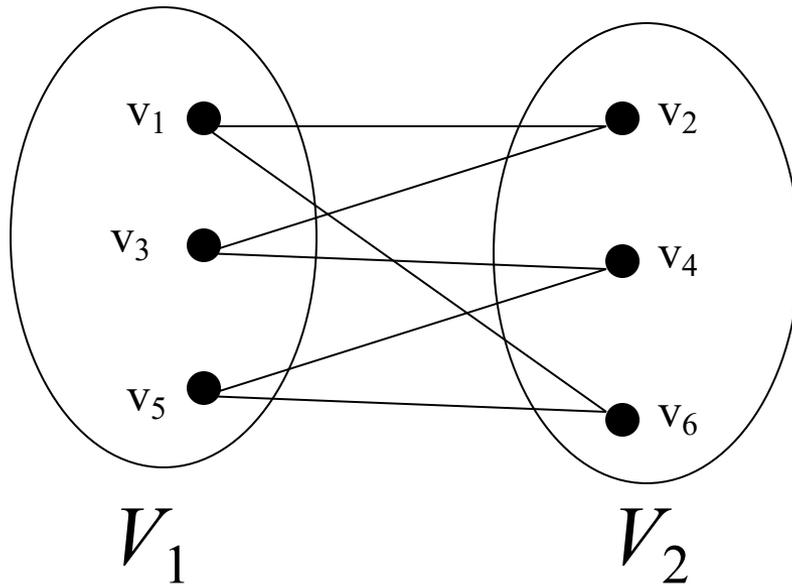
- The  $n$ -dimensional hypercube, or  $n$ -cube, denoted by  $Q_n$ , is the graph that has vertices representing the  $2^n$  bit strings of length  $n$ . Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.



Note.  $Q_n$  is  $n$ -regular,  $|V(Q_n)| = 2^n$ ,  $|E(Q_n)| = (2^n n)/2 = 2^{n-1} n$

# Some Special Simple Graphs: Bipartite

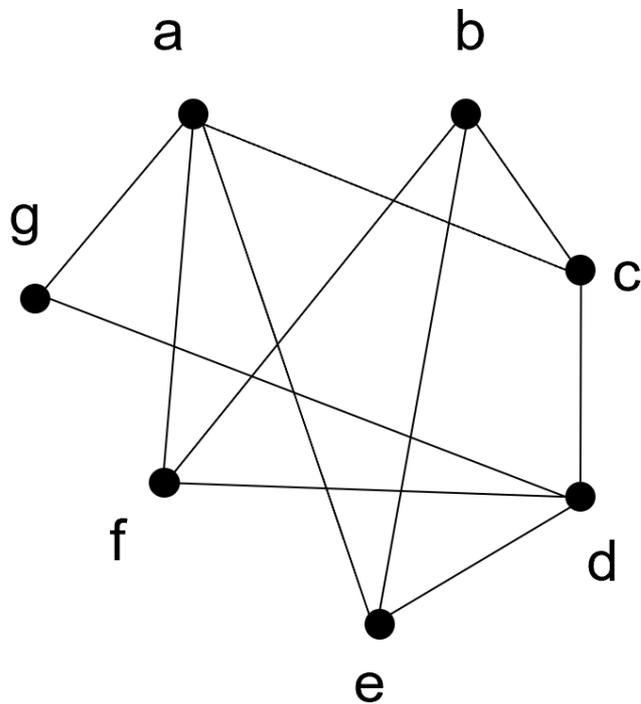
- A simple graph  $G=(V,E)$  is called **bipartite** if  $V$  can be partitioned into  $V_1$  and  $V_2$ ,  $V_1 \cap V_2 = \emptyset$ , such that every edge in the graph connects a vertex in  $V_1$  and a vertex in  $V_2$ .



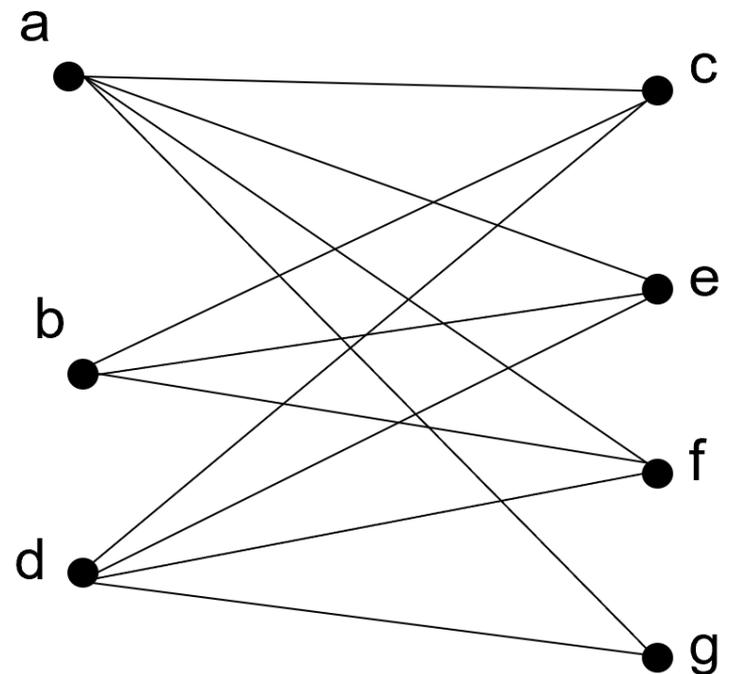
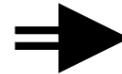
$\therefore C_6$  is bipartite.

# Some Special Simple Graphs: Bipartite

- Is the graph  $G$  bipartite ?



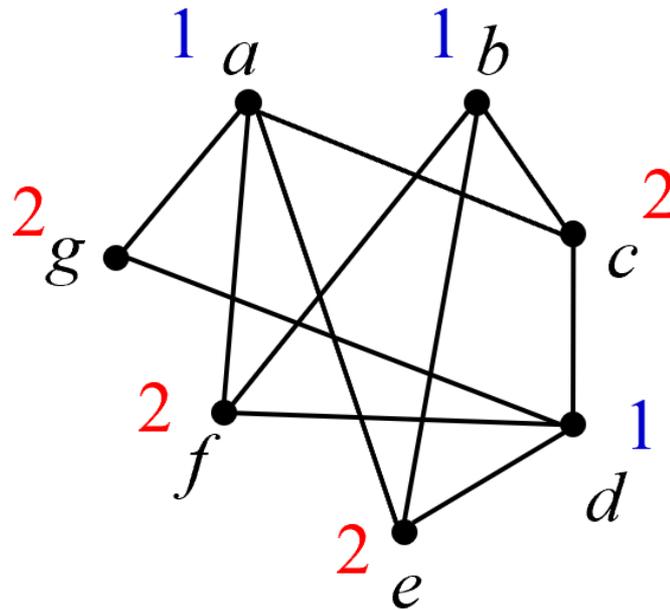
$G$



**Yes !**

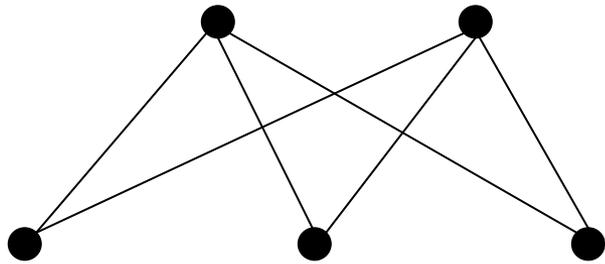
# Some Special Simple Graphs: Bipartite

- A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

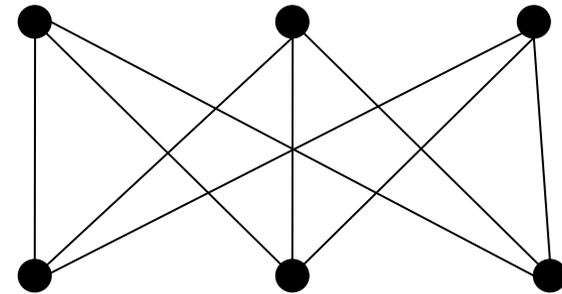


# Some Special Simple Graphs: Bipartite

- Complete Bipartite graphs ( $K_{m,n}$ )



$K_{2,3}$

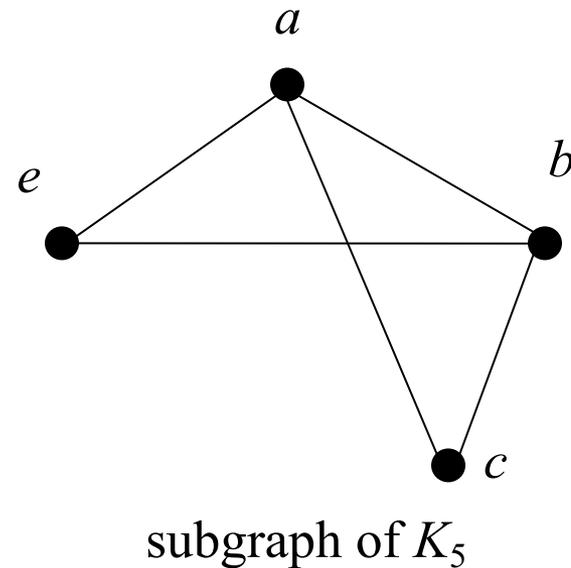
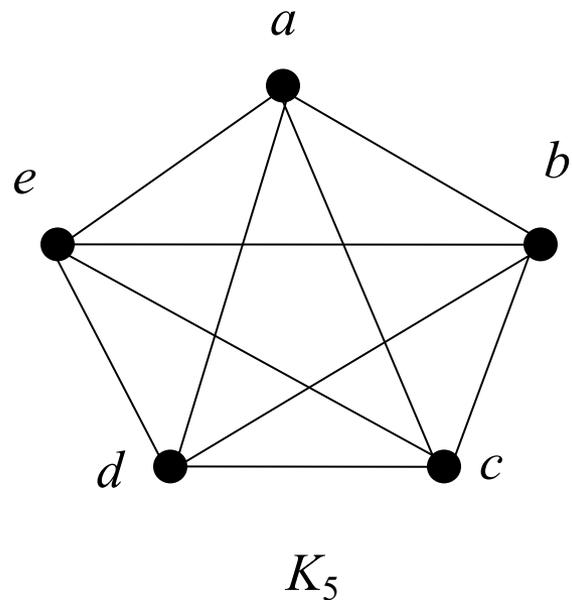


$K_{3,3}$

Note.  $|V(K_{m,n})| = m+n$ ,  $|E(K_{m,n})| = mn$ ,  
 $K_{m,n}$  is regular if and only if  $m=n$ .

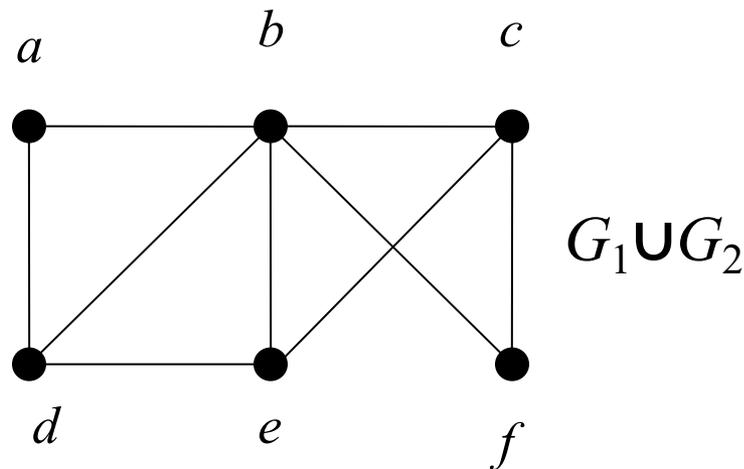
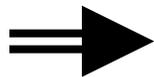
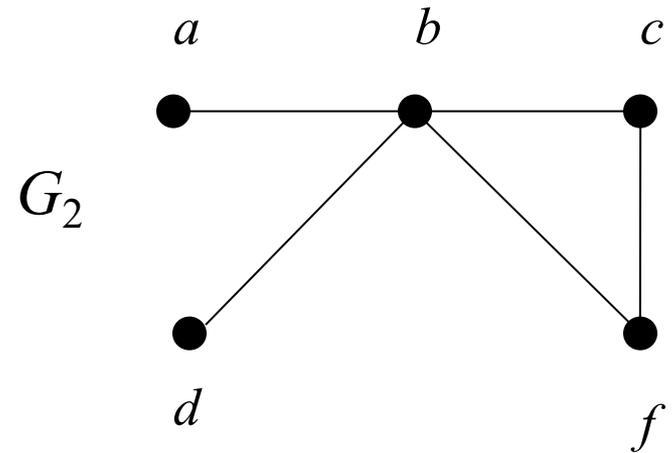
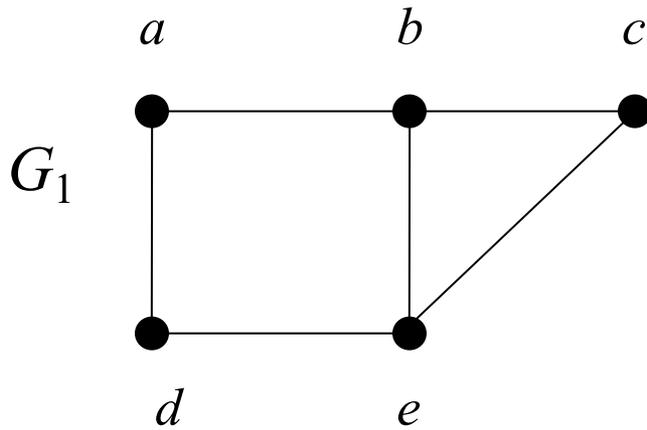
# New Graphs from Old

- A **subgraph** of a graph  $G=(V, E)$  is a graph  $H=(W, F)$  where  $W \subseteq V$  and  $F \subseteq E$ . (Notice the f point w to connect)



# The union of two simple graphs

- $G_1=(V_1, E_1)$  and  $G_2=(V_2, E_2)$  is the simple graph  
 $G_1 \cup G_2=(V_1 \cup V_2, E_1 \cup E_2)$



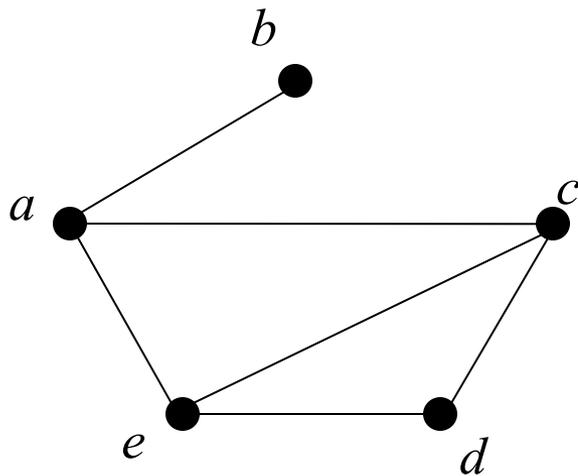
# Outline

- Graph and Graph Models
- Graph Terminology and Special Types of Graphs
- **Representing Graphs and Graph Isomorphism**

# Representing Graphs and Graph Isomorphism

## Adjacency list: Undirected graph

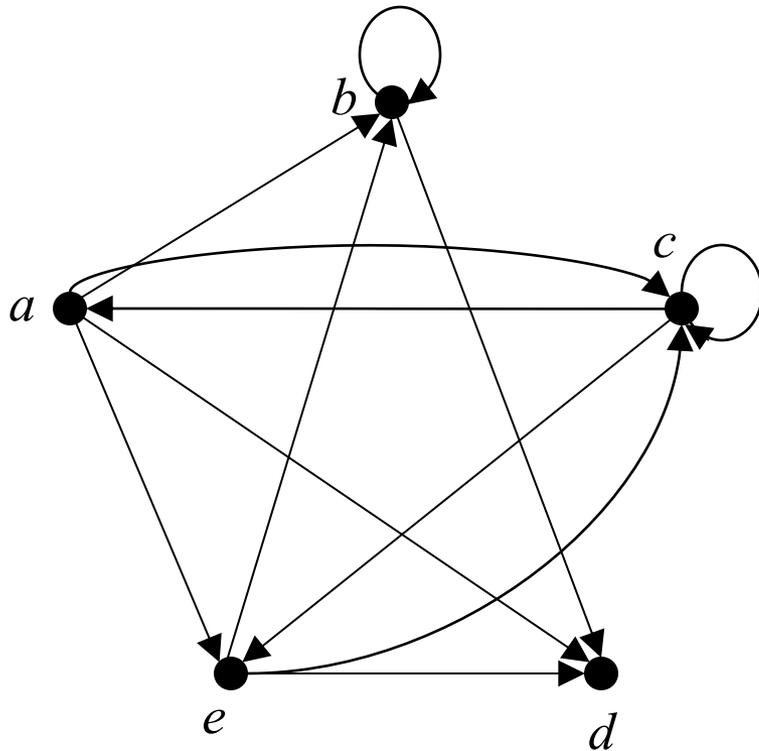
**Example:** Use adjacency lists to describe the simple graph given below.



Vertex	Adjacent Vertices
$a$	$b,c,e$
$b$	$a$
$c$	$a,d,e$
$d$	$c,e$
$e$	$a,c,d$

# Representing Graphs and Graph Isomorphism

- **Adjacency list: Directed graph**



Initial vertex	Terminal vertices
<i>a</i>	<i>b, c, d, e</i>
<i>b</i>	<i>b, d</i>
<i>c</i>	<i>a, c, e</i>
<i>d</i>	
<i>e</i>	<i>b, c, d</i>

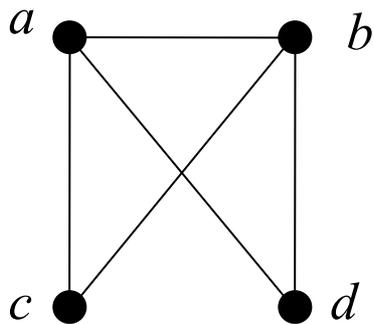
# Representing Graphs and Graph Isomorphism

## Adjacency Matrices

$G=(V, E)$  : simple graph,  $V=\{v_1, v_2, \dots, v_n\}$ .

A matrix  $A$  is called the **adjacency matrix** of  $G$

if  $A=[a_{ij}]_{n \times n}$ , where  $a_{ij} = 1$ , if  $\{v_i, v_j\} \in E$ , and 0 otherwise.



$$A_1 = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$A_2 = \begin{matrix} & \begin{matrix} b & d & c & a \end{matrix} \\ \begin{matrix} b \\ d \\ c \\ a \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

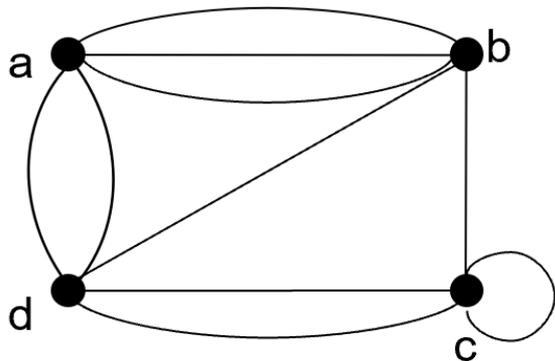
### Note:

1. There are  $n!$  different adjacency matrices for a graph with  $n$  vertices.
2. The adjacency matrix of an undirected graph is **symmetric**.
3.  $a_{ii} = 0$  (simple matrix has no loop)

# Representing Graphs and Graph Isomorphism

## Adjacency Matrices

(Pseudograph) (Matrix may not be 0,1 matrix.)



$$A = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{matrix}$$

If  $A=[a_{ij}]$  is the adjacency matrix for the directed graph, then

$$a_{ij} = \begin{cases} 1 & , \text{ if } \begin{matrix} \bullet & \longrightarrow & \bullet \\ v_i & & v_j \end{matrix} \\ 0 & , \text{ otherwise} \end{cases}$$

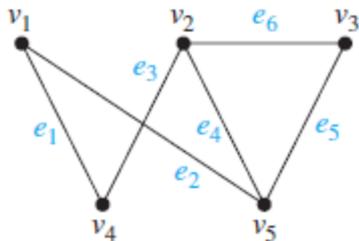
(So the matrix is not necessarily symmetrical)

# Representing Graphs and Graph Isomorphism

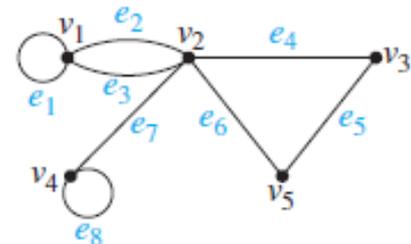
- Incidence Matrices**

- Let  $G=(V, E)$  : be an undirected graph. Suppose that  $v_1, v_2, \dots, v_n$  are the vertices and  $e_1, e_2, \dots, e_n$  are the edges of  $G$  . Then the incidence matrix with respect to this ordering of  $V$  and  $E$  is the  $n \times m$  matrix  $M=[m_{ij}]$ , where

$$m_{i,j} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$



	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$
$v_1$	1	1	0	0	0	0
$v_2$	0	0	1	1	0	1
$v_3$	0	0	0	0	1	1
$v_4$	1	0	1	0	0	0
$v_5$	0	1	0	1	1	0

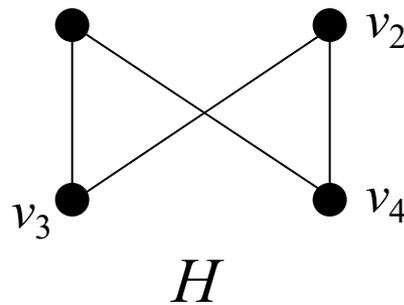
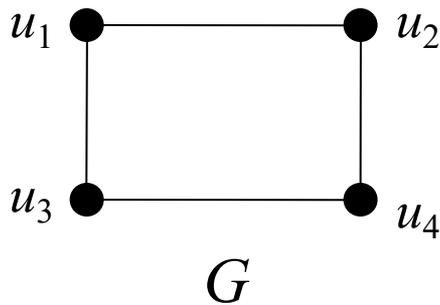


	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$	$e_8$
$v_1$	1	1	1	0	0	0	0	0
$v_2$	0	1	1	1	0	1	1	0
$v_3$	0	0	0	1	1	0	0	0
$v_4$	0	0	0	0	0	0	1	1
$v_5$	0	0	0	0	1	1	0	0

# Representing Graphs and Graph Isomorphism

- **Isomorphism of Graphs**

The simple graphs  $G_1=(V_1,E_1)$  and  $G_2=(V_2,E_2)$  are **isomorphic** if there is an one-to-one and onto function  $f$  from  $V_1$  to  $V_2$  with the property that  $a \sim b$  in  $G_1$  iff  $f(a) \sim f(b)$  in  $G_2$ ,  $\forall a,b \in V_1$ ,  $f$  is called an **isomorphism**.

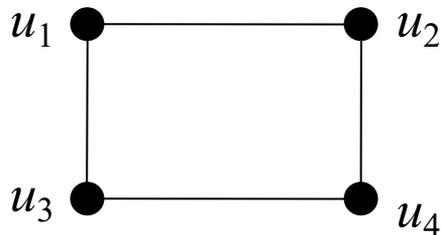


$G$  is isomorphic to  $H$

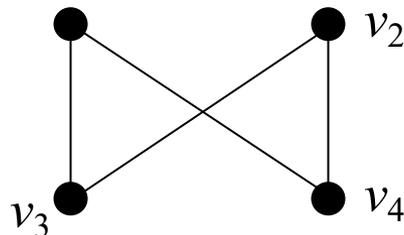
# Representing Graphs and Graph Isomorphism

- Show that  $G$  and  $H$  are isomorphic.

**Solution:** The function  $f$  with  $f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3,$  and  $f(u_4) = v_2$  is a one-to-one correspondence between  $V(G)$  and  $V(H)$ .



$G$



$H$

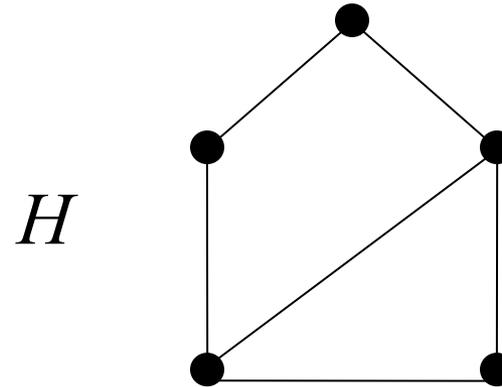
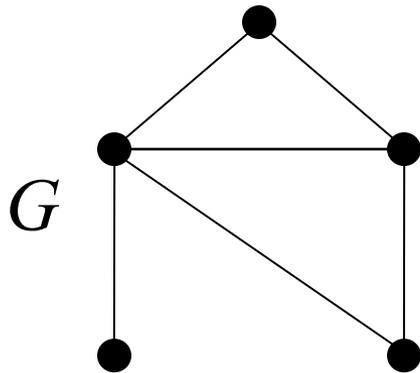
$G$  is isomorphic to  $H$

Isomorphism graphs  
there will be:

1. The same number of points (vertices)
2. The same number of edges
3. The same number of degree

# Representing Graphs and Graph Isomorphism

- Show that  $G$  and  $H$  are not isomorphic.

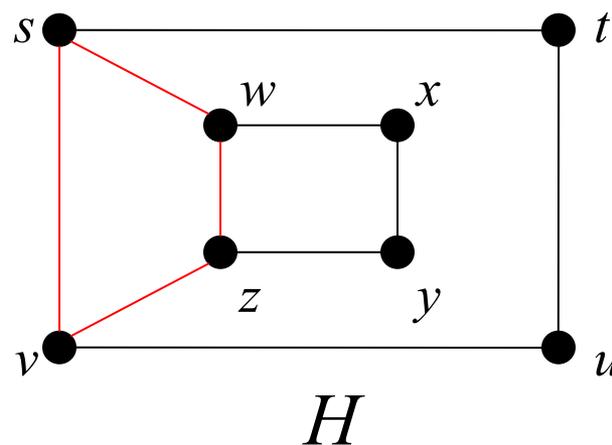
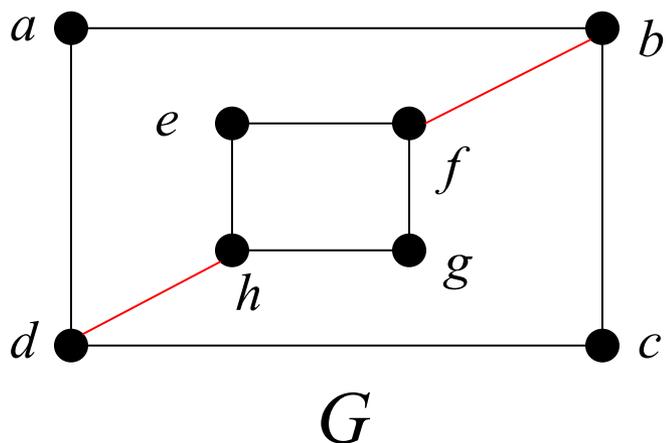


## Solution :

$G$  has a vertex of degree = 1 ,  $H$  don't

# Representing Graphs and Graph Isomorphism

- Determine whether  $G$  and  $H$  are isomorphic.



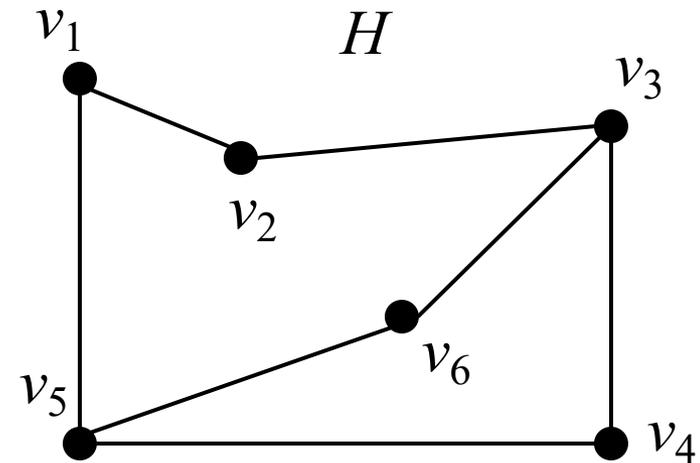
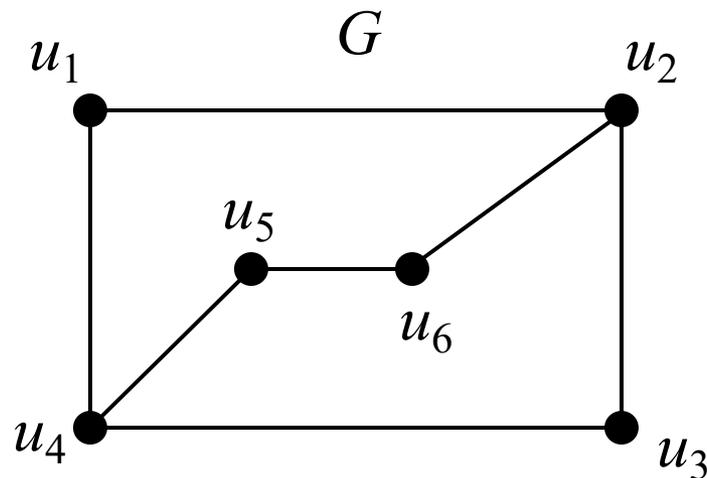
**Solution :**  $\because$  In  $G$ ,  $\text{deg}(a)=2$ , which must correspond to either  $t$ ,  $u$ ,  $x$ , or  $y$  in  $H$  degree

Each of these four vertices in  $H$  is adjacent to another vertex of degree two in  $H$ , which is not true for  $a$  in  $G$

$\therefore G$  and  $H$  are not isomorphic.

# Representing Graphs and Graph Isomorphism

- Determine whether the graphs  $G$  and  $H$  are isomorphic.



## Solution:

$$f(u_1)=v_6, f(u_2)=v_3, f(u_3)=v_4, f(u_4)=v_5, f(u_5)=v_1, f(u_6)=v_2$$

$\Rightarrow$  Yes

# Next class

- Topic: Graph Theory (2)
- Pre-class reading: Chap 10.3-10.5

