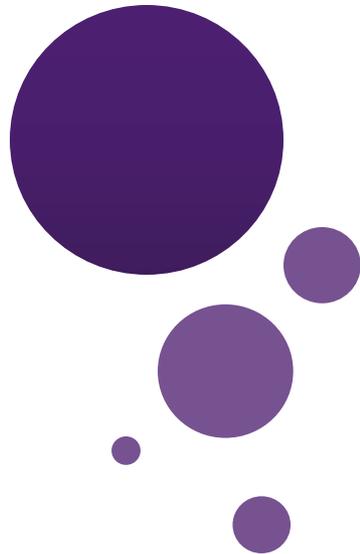




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State University of New York

Lecture 34: Graph Theory (3)



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Recap Previous Lecture

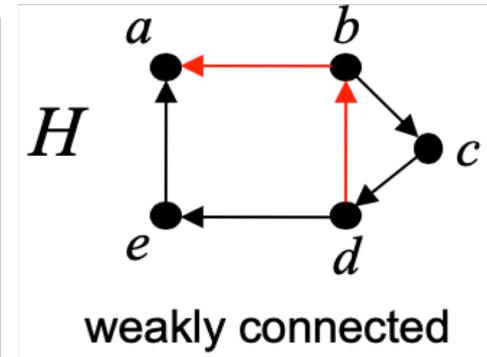
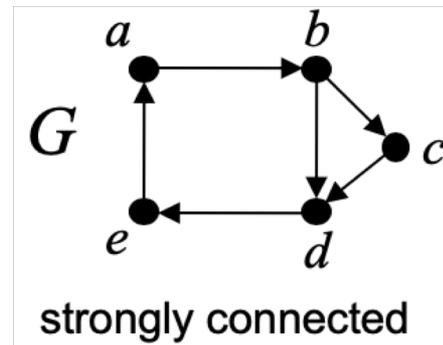
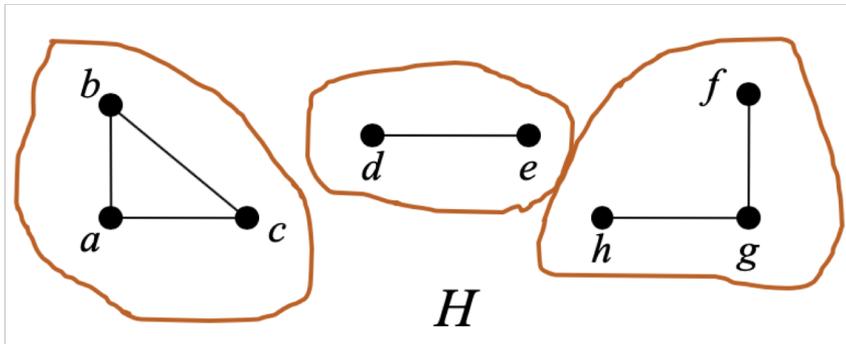
- Representing Graphs and Graph Isomorphism
- Connectivity

Vertex	Adjacent Vertices
<i>a</i>	<i>b,c,e</i>
<i>b</i>	<i>a</i>
<i>c</i>	<i>a,d,e</i>
<i>d</i>	<i>c,e</i>
<i>e</i>	<i>a,c,d</i>

Initial vertex	Terminal vertices
<i>a</i>	<i>b,c,d,e</i>
<i>b</i>	<i>b,d</i>
<i>c</i>	<i>a,c,e</i>
<i>d</i>	
<i>e</i>	<i>b,c,d</i>

$$A = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$



Outline

- Euler and Hamilton Paths
- Shortest-Path Problems
- Planar Graphs
- Graph Coloring

Outline

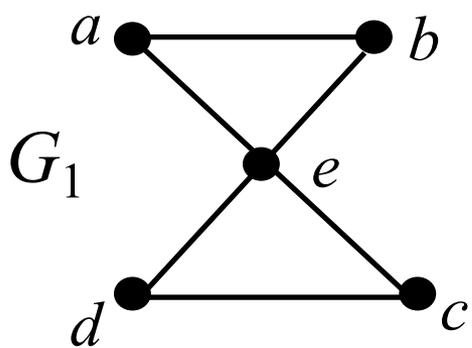
- **Euler and Hamiltonian Paths**
- Shortest-Path Problems
- Planar Graphs
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Euler and Hamilton Paths

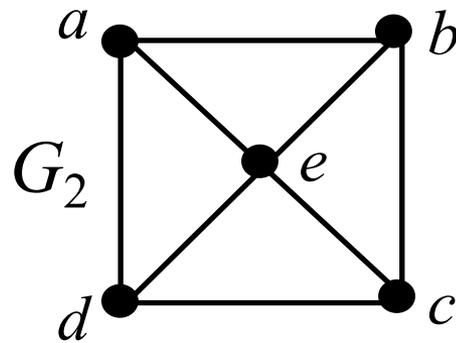
- An *Euler circuit* in a graph G is a simple circuit containing every edge of G .
An *Euler path* in G is a simple path containing every edge of G .
- A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.
- A connected multigraph has an Euler path (**but not an Euler circuit**) if and only if it has exactly 2 vertices of odd degree.

Example

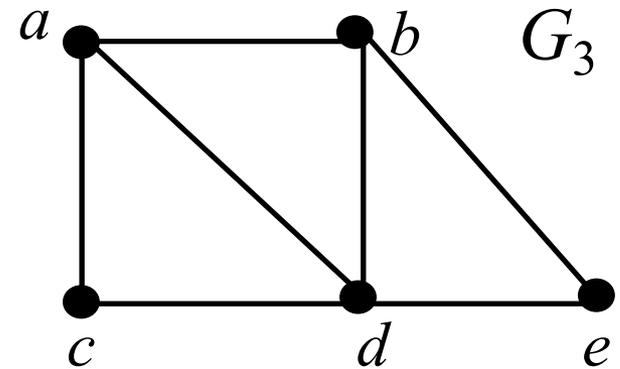
- Which of the following graphs have an Euler circuit or an Euler path?



Euler circuit



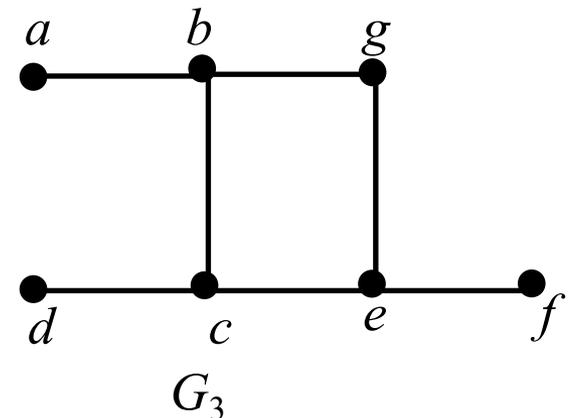
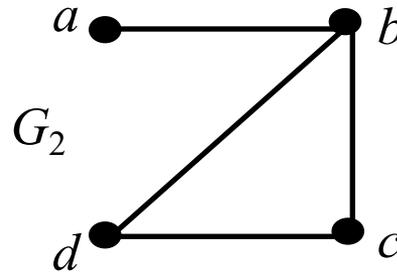
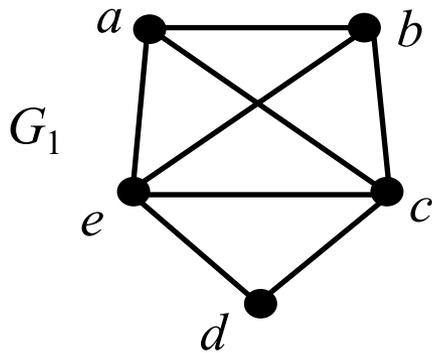
none



Euler path

Hamilton Paths and Circuits

- A *Hamilton path* is a path that traverses each vertex in a graph G exactly once.
A *Hamilton circuit* is a circuit that traverses each vertex in G exactly once.
- Which of the following graphs have a Hamilton circuit or a Hamilton path?



Hamilton circuit: G_1

Hamilton path: G_1, G_2

Outline

- Euler and Hamiltonian Paths
- **Shortest-Path Problems**
- Planar Graphs
- Graph Coloring

Shortest-Path Problems

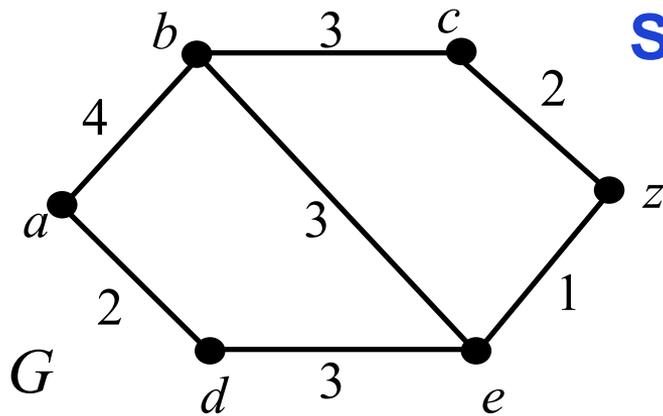
- Graphs that have a number assigned to each edge are called *weighted graphs*.
- The **length** of a path in a weighted graph is the sum of the weights of the edges of this path.

Shortest path Problem:

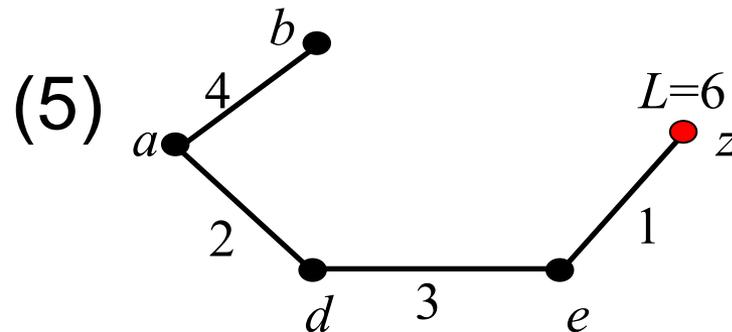
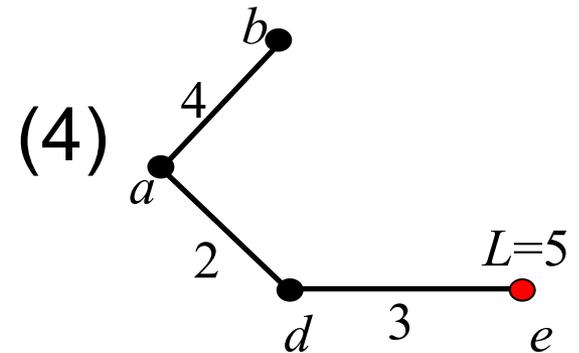
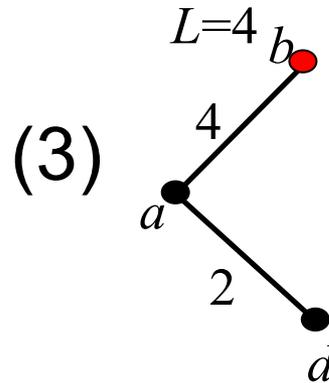
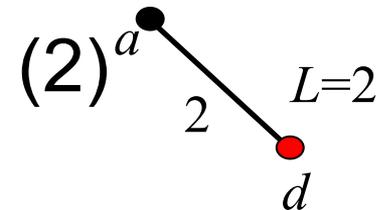
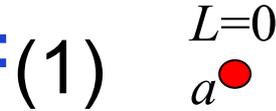
Determining the path of least sum of the weights between two vertices in a weighted graph.

Exampe

- What is the length of a shortest path between a and z in the weighted graph G ?



Solution:



length=6

Dijkstra's Algorithm

Procedure *Dijkstra*(G : weighted connected simple graph,
with all weights positive)

{ G has vertices $a = v_0, v_1, \dots, v_n = z$ and weights $w(v_i, v_j)$
where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G }

for $i := 1$ **to** n

$L(v_i) := \infty$

$L(a) := 0$

$S := \emptyset$

while $z \notin S$

begin

$u :=$ a vertex not in S with $L(u)$ minimal

$S := S \cup \{u\}$

for all vertices v not in S

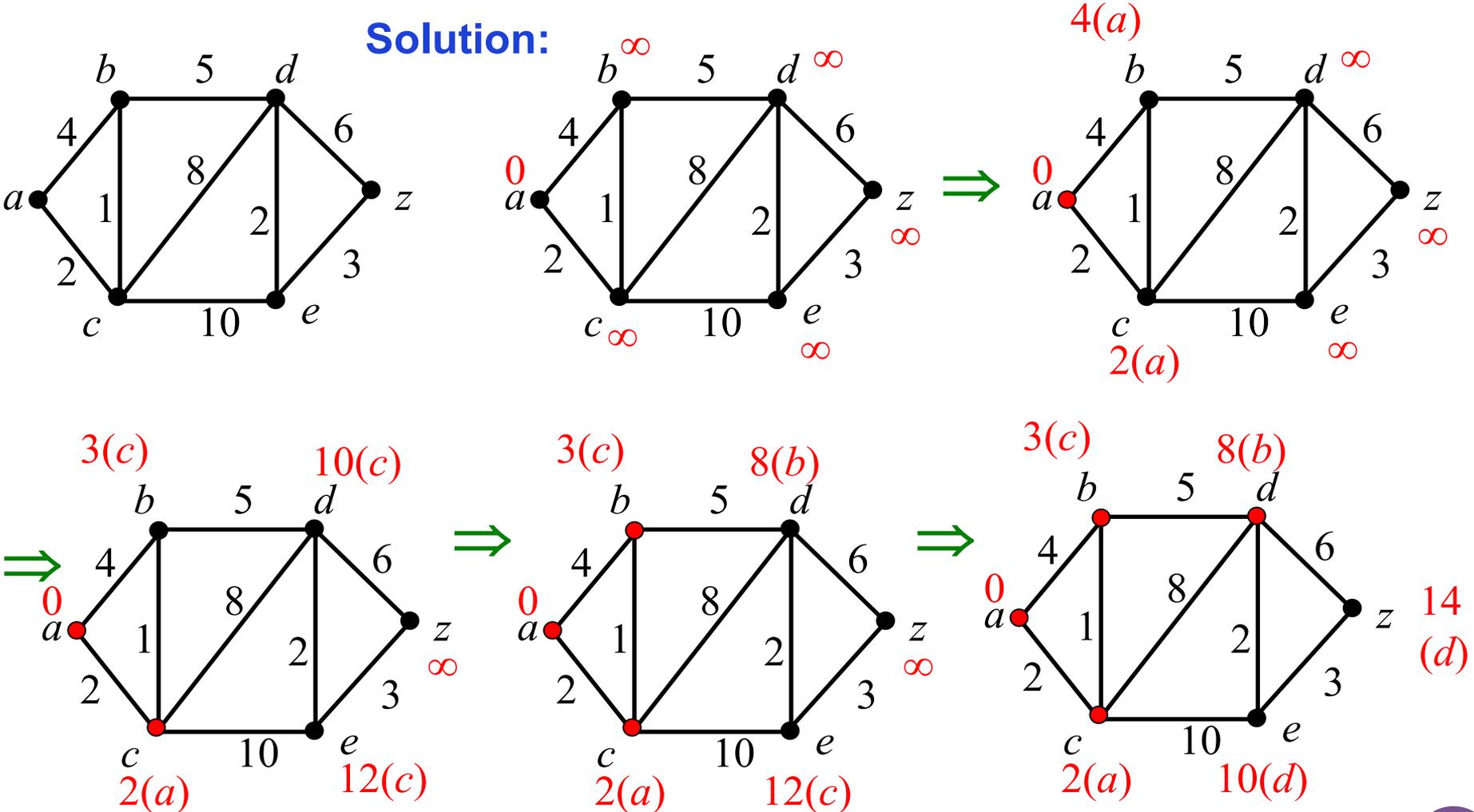
if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$

end { $L(z) =$ length of a shortest path from a to z }

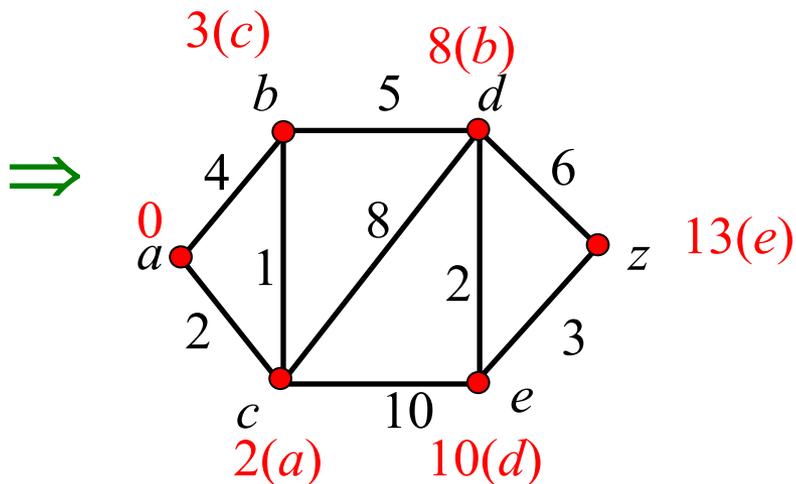
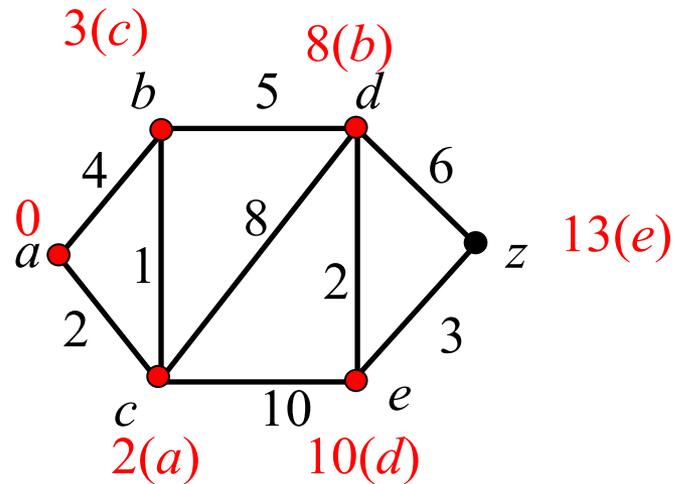
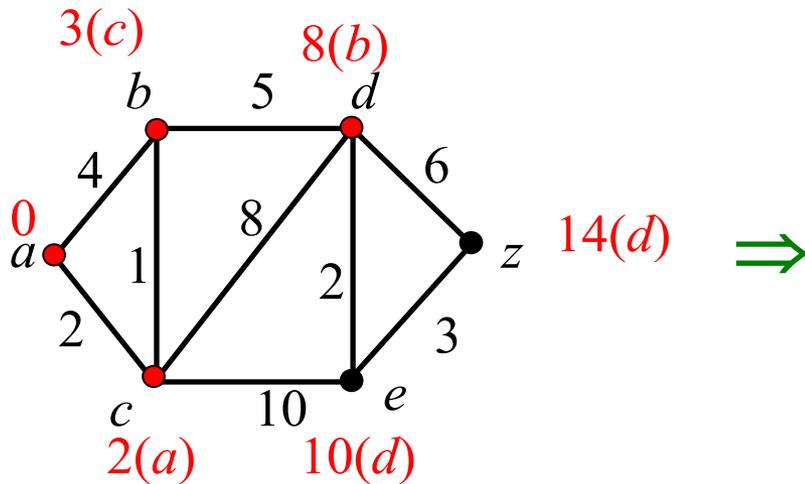
Example

- Use Dijkstra's algorithm a shortest path between a and z .

Solution:



Example: Cont.



⇒ path: a, c, b, d, e, z
length: 13

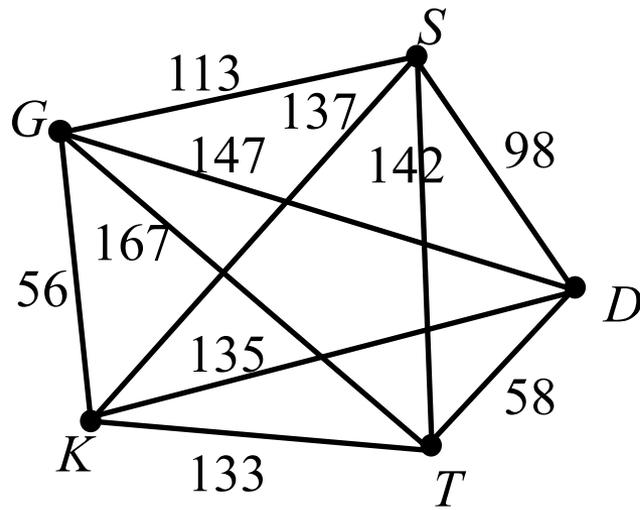
Remarks on Dijkstra's Algorithm

- Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.
- Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with n vertices.

The Traveling Salesman Problem

- A traveling salesman wants to visit each of n cities exactly once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?

Example (starting point D)



$$D \rightarrow T \rightarrow K \rightarrow G \rightarrow S \rightarrow D: 458$$

$$D \rightarrow T \rightarrow S \rightarrow G \rightarrow K \rightarrow D: 504$$

$$D \rightarrow T \rightarrow S \rightarrow K \rightarrow G \rightarrow D: 540$$

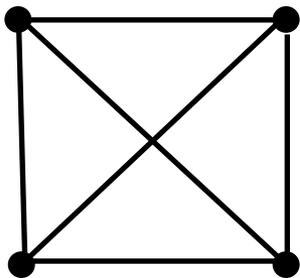
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Outline

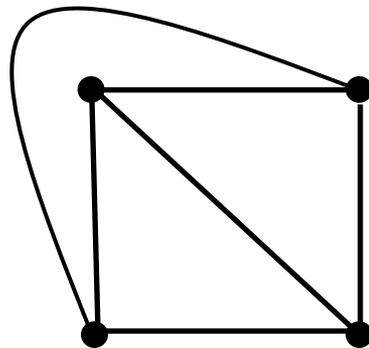
- Euler and Hamiltonian Paths
- Shortest-Path Problems
- **Planar Graphs**
- Graph Coloring

Planar Graphs

- A graph is called *planar* if it can be drawn in the plane without any edge crossing. Such a drawing is called a *planar representation* of the graph.
- Is K_4 planar?



K_4

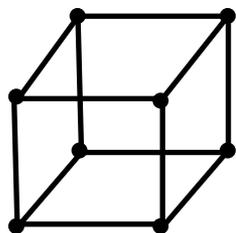


K_4 drawn with
no crossings

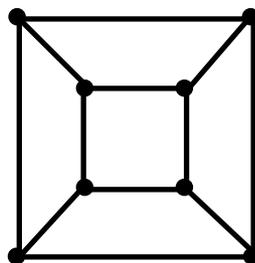
$\therefore K_4$ is planar

Example

- Is Q_3 planar?



Q_3

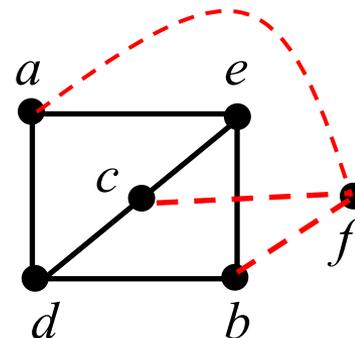
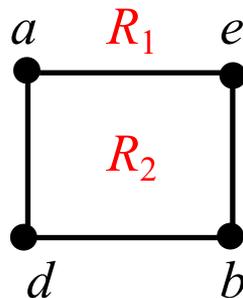
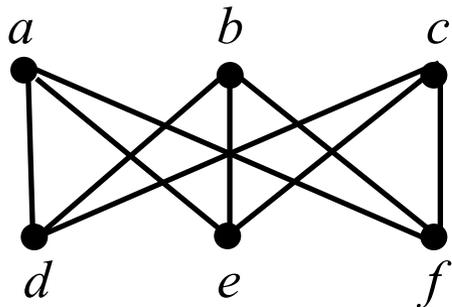


Q_3 drawn with no crossings

$\therefore Q_3$ is planar

- Show that $K_{3,3}$ is nonplanar.

Solution:

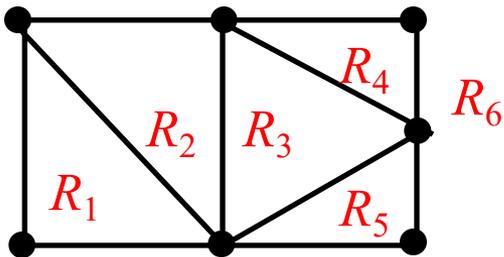


In any drawing, $aebd$ is cycle, and will cut the plane into two regions

Regardless of which region c , could no longer put the f in that side staggered

Euler's Formula

- A planar representation of a graph splits the plane into **regions**, including an unbounded region.
- **Example** : How many regions are there in the following graph?



Solution: 6

Euler's Formula

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G . Then $r = e - v + 2$.

Example

- Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

- **Solution:**

$$v = 20, 2e = 3 \times 20 = 60, e = 30$$

$$r = e - v + 2 = 30 - 20 + 2 = 12$$

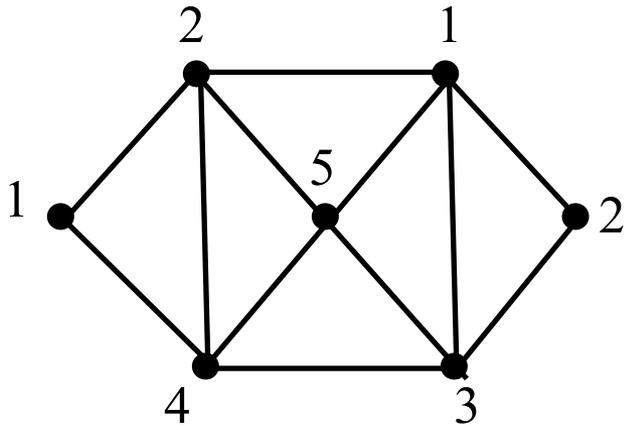
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- Euler and Hamiltonian Paths
- Shortest-Path Problems
- Planar Graphs
- **Graph Coloring**

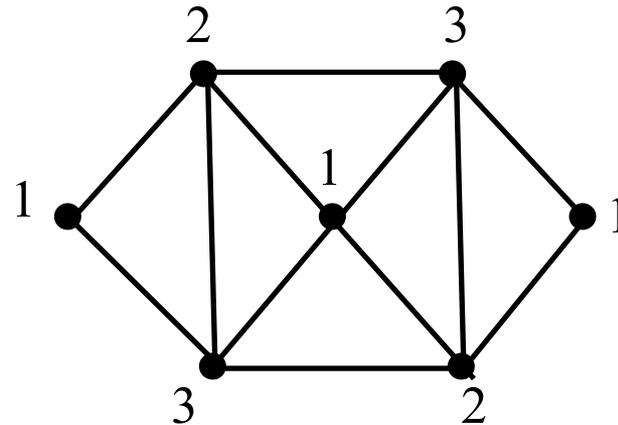
Graph Coloring

- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Example:



5-coloring



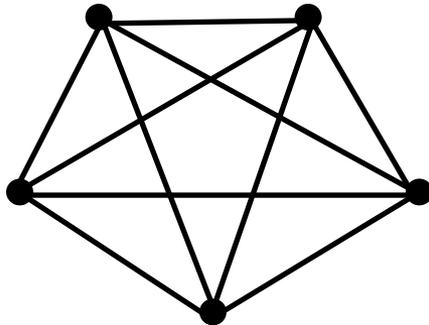
3-coloring

Less the number of colors, the better

Graph Coloring

- The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph. (denoted by $\chi(G)$)

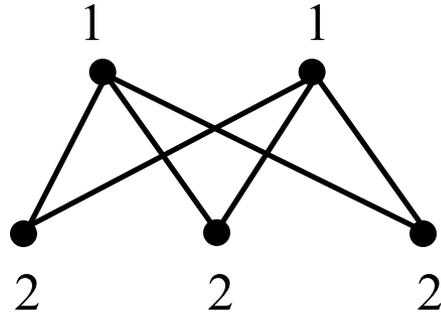
Example: $\chi(K_5)=5$



Note: $\chi(K_n)=n$

Example

$$\chi(K_{2,3}) = 2.$$

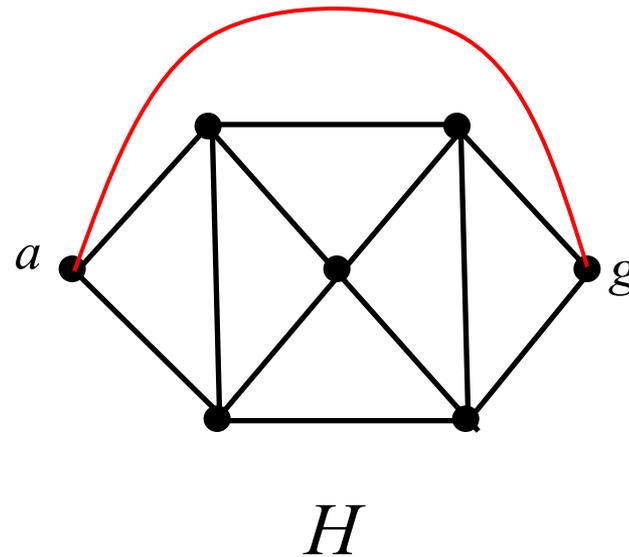
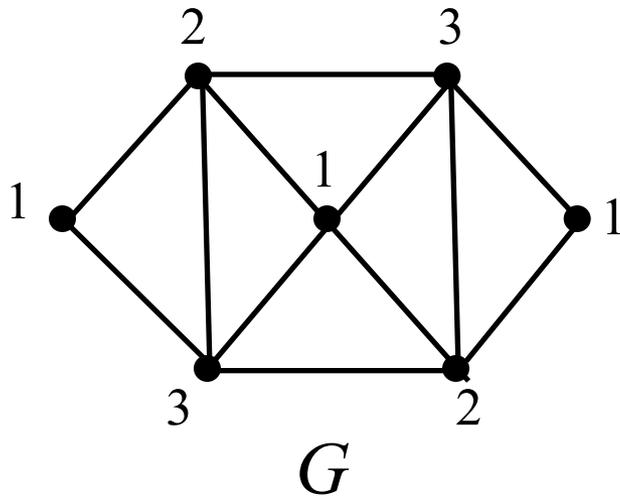


Note: $\chi(K_{m,n}) = 2$

Note: If G is a bipartite graph, $\chi(G) = 2$.

Example

- What are the chromatic numbers of the graphs G and H ?



Solution: G has a 3-cycle

$$\Rightarrow \chi(G) \geq 3$$

G has a 3-coloring

$$\Rightarrow \chi(G) \leq 3$$

$$\Rightarrow \chi(G) = 3$$

Solution: any 3-coloring for

$H - \{(a,g)\}$ gives the

same color to a and g

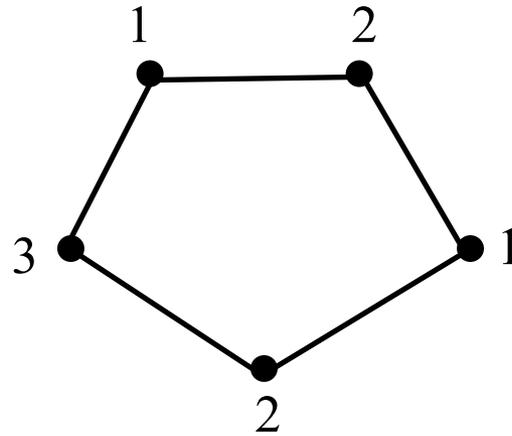
$$\Rightarrow \chi(H) > 3$$

4-coloring exists $\Rightarrow \chi(H) = 4$

Example

$\chi(C_n) = 2$ if n is even, $\chi(C_n) = 3$ if n is odd.

C_n is bipartite
when n is even.



Theorem 1. (The Four Color Theorem)

The chromatic number of a planar graph is no greater than four.

Corollary

Any graph with chromatic number >4 is nonplanar.

Next class

- Topic: Trees
- Pre-class reading: Chap 11

