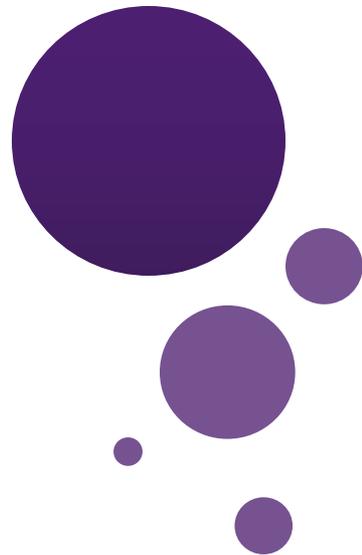




UNIVERSITY
AT ALBANY

State University of New York

Lecture 4: Quantifiers and Rules of Inference



Dr. Chengjiang Long
Computer Vision Researcher at Kitware Inc.
Adjunct Professor at SUNY at Albany.
Email: clong2@albany.edu

Recap Previous Lecture

1. Logical Equivalence with Known Laws
2. Predicate Logical
3. Quantifiers and Nested Quantifiers

Statement	True when...	False when...
$\forall x P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false
$\exists x P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x

Statement	True when...	False when...
$\forall x \forall y P(x,y)$	$P(x,y)$ is true for every pair x,y	There is at least one <i>pair</i> x,y for which $P(x,y)$ is false
$\forall x \exists y P(x,y)$	For every x , there is a y for which $P(x,y)$ is true	There is an x for which $P(x,y)$ is false for every y
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y	For every x , there is a y for which $P(x,y)$ is false
$\exists x \exists y P(x,y)$	There is at least one pair x,y for which $P(x,y)$ is true	$P(x,y)$ is false for every pair x,y

Outline

- Quantifiers
- Logic Programming (Prolog)
- Transcribing English to Logic
- Rules of Inference

Outline

- **Quantifiers**
- Logic Programming (Prolog)
- Transcribing English to Logic
- Rules of Inference

Mixing Quantifiers: Example (1)

- Express, in predicate logic, the statement that there is an infinite number of integers
- Answer:
 - Let $P(x,y)$ be the statement that $x < y$
 - Let the universe of discourse be the integers, \mathbb{Z}
 - The statement can be expressed by the following

$$\forall x \exists y P(x,y)$$

Mixing Quantifiers: Example (2)

- Express the *commutative law of addition* for \mathcal{R}
- We want to express that for every pair of reals, x, y , the following holds: $x+y=y+x$
- Answer:

– Let $P(x, y)$ be the statement that $x+y$

– Let the universe of discourse be the reals, \mathcal{R}

– The statement can be expressed by the following

$$\forall x \forall y (P(x, y) \Leftrightarrow P(y, x))$$

Alternatively, $\forall x \forall y (x+y = y+x)$

Mixing Quantifiers: Example (3)

- Express the multiplicative *law* for nonzero reals $\mathbb{R} \setminus \{0\}$
- We want to express that for every real number x , there exists a real number y such that $xy=1$
- Answer:

$$\forall x \exists y (xy = 1)$$

Mixing Quantifiers: Example (4)

false mathematical statement

- Does commutativity for subtraction hold over the reals?
- That is: does $x-y=y-x$ for all pairs x,y in \mathbb{R} ?
- Express using quantifiers

$$\forall x \forall y (x-y = y-x)$$

Mixing Quantifiers: Example (5)

- Express the statement as a logical expression: “There is a number x such that when it is added to any number, the result is that number and if it is multiplied by any number, the result is x ” as a logical expression
- Answer:
 - Let $P(x,y)$ be the expression “ $x+y=y$ ”
 - Let $Q(x,y)$ be the expression “ $xy=x$ ”
 - The universe of discourse is N, Z, R, Q (but not Z^+)
 - Then the expression is:

$$\exists x \forall y P(x,y) \wedge Q(x,y)$$

Alternatively: $\exists x \forall y (x+y=y) \wedge (xy = x)$

Binding Variables

- When a quantifier is used on a variable x , we say that x is bound
- If no quantifier is used on a variable in a predicate statement, the variable is called free
- Examples
 - In $\exists x \forall y P(x, y)$, both x and y are bound
 - In $\forall x P(x, y)$, x is bound but y is free
- A statement is called a well-formed formula, when all variables are properly quantified

Binding Variables: Scope

- The set of all variables bound by a common quantifier is called the scope of the quantifier
- For example, in the expression $\exists x, y \forall z P(x, y, z, c)$
 - What is the scope of existential quantifier?
 - What is the scope of universal quantifier?
 - What are the bound variables?
 - What are the free variables?
 - Is the expression a well-formed formula?

Negation

- We can use negation with quantified expressions as we used them with propositions
- **Lemma:** Let $P(x)$ be a predicate. Then the followings hold:

$$\neg(\forall x P(x)) \equiv \exists x \neg P(x)$$

$$\neg(\exists x P(x)) \equiv \forall x \neg P(x)$$

- This is essentially the quantified version of De Morgan's Law (when the universe of discourse is finite, this is exactly De Morgan's Law)

Negation: Truth

Truth Values of Negated Quantifiers

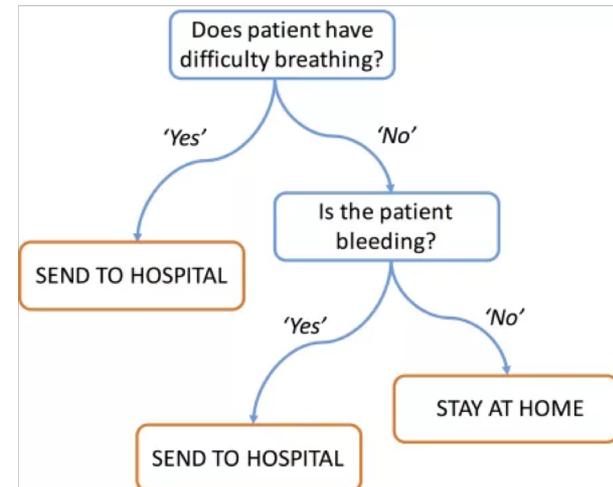
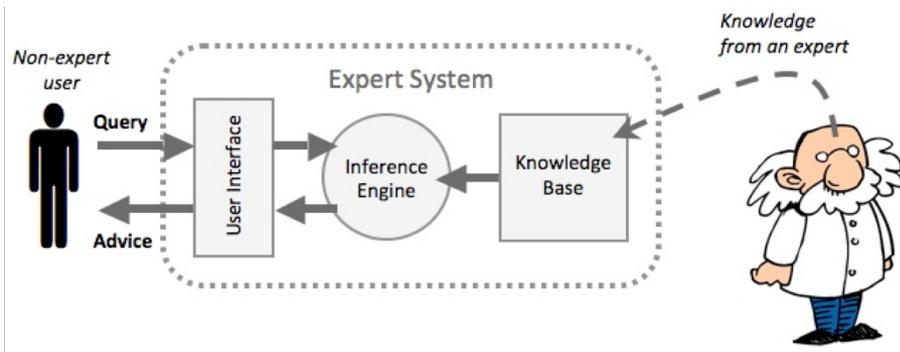
Statement	True when...	False when...
$\neg \exists x P(x) \equiv \forall x \neg P(x)$	$P(x)$ is false for every x	There is an x for which $P(x)$ is true
$\neg \forall x P(x) \equiv \exists x \neg P(x)$	There is an x for which $P(x)$ is false	$P(x)$ is true for every x

Outline

- Quantifiers
- **Logic Programming (Prolog)**
- Transcribing English to Logic
- Rules of Inference

Prolog (1)

- Prolog (Programming in Logic) is a programming language based on (a restricted form of) Predicate Logic (a.k.a. Predicate Calculus and FOL)
- It was developed by the logicians of the Artificial Intelligence community for symbolic reasoning



Prolog (2)

- Prolog allows the users to express facts and rules
- Facts are propositional functions: `student(mia)`, `enrolled(mia,cse235)`, `instructor(patel,cse235)`, etc.
- Rules are implications with conjunctions:
`teaches(X,Y) :- instructor(X,Z), enrolled(Y,Z)`
- Prolog answers queries such as:
`?enrolled(mia,cse235)`
`?enrolled(X,cse476)`
`?teaches(X,mia)`
by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5

Outline

- Quantifiers
- Logic Programming (Prolog)
- **Transcribing English to Logic**
- Rules of Inference

English into Logic

- Logic is more precise than English
- Transcribing English into Logic and vice versa can be tricky
- When writing statements with quantifiers, usually the correct meaning is conveyed with the following combinations:

Use \forall with \Rightarrow

$\forall x \text{ Lion}(x) \Rightarrow \text{Fierce}(x)$: Every lion is fierce

$\forall x \text{ Lion}(x) \wedge \text{Fierce}(x)$: Everyone is a lion and everyone is fierce

Use \exists with \wedge

$\exists x \text{ Lion}(x) \wedge \text{Vegan}(x)$: Holds when you have at least one vegan lion

$\exists x \text{ Lion}(x) \Rightarrow \text{Vegan}(x)$: Holds when you have vegan people in the universe of discourse (even though there is no vegan lion in the universe of discourse)

More exercises (1)

- Rewrite the following expression, pushing negation inward:

$$\neg \forall x (\exists y \forall z P(x,y,z) \wedge \exists z \forall y P(x,y,z))$$

- Answer:

$$\exists x (\forall y \exists z \neg P(x,y,z) \vee \forall z \exists y \neg P(x,y,z))$$

More Exercises (2)

- Let $P(x,y)$ denote 'x is a factor of y' where
 - $x \in \{1,2,3,\dots\}$ and $y \in \{2,3,4,\dots\}$
- Let $Q(x,y)$ denote:
 - $\forall x,y [P(x,y) \rightarrow (x=y) \vee (x=1)]$
- Question: When is $Q(x,y)$ true?

Alert...

- Some students wonder if:

$$\forall x,y P(x,y) \equiv (\forall x P(x,y)) \wedge (\forall y P(x,y))$$

- This is certainly not true.
 - In the left-hand side, both x,y are bound.
 - In the right-hand side,
 - In the first predicate, x is bound and y is free
 - In the second predicate, y is bound and x is free
 - Thus, the left-hand side is a proposition, but the right-hand side is not. They cannot be equivalent
- All variables that occur in a propositional function must be bound to turn it into a proposition

Outline

- Quantifiers
- Logic Programming (Prolog)
- Transcribing English to Logic
- **Rules of Inference**

Motivation (1)

- “Mathematical proofs, like diamonds, are hard and clear, and will be touched with nothing but strict reasoning.” *-John Locke*
- Mathematical proofs are, in a sense, the only true knowledge we have
- They provide us with a guarantee as well as an explanation (and hopefully some insight)

Motivation (2)

- Mathematical proofs are necessary in CS
 - You must always (try to) prove that your algorithm
 - terminates
 - is sound, complete, optimal
 - finds optimal solution
 - You may also want to show that it is more efficient than another method
 - Proving certain properties of data structures may lead to new, more efficient or simpler algorithms
 - Arguments may entail assumptions. You may want to prove that the assumptions are valid

Concepts

- A **theorem** is a statement that can be shown to be true (via a proof)
- A **proof** is a sequence of statements that form an argument
- A **corollary** is a theorem that can be established from theorem that has just been proven
- A **conjecture** is a statement whose truth value is unknown
- The **rules of inference** are the means used to draw conclusions from other assertions, and to derive an argument or a proof

Theorems: Example

- Theorem
 - Let a , b , and c be integers. Then
 - If $a|b$ and $a|c$ then $a|(b+c)$
 - If $a|b$ then $a|bc$ for all integers c
 - If $a|b$ and $b|c$, then $a|c$
- Corollary:
 - If a , b , and c are integers such that $a|b$ and $a|c$, then $a|mb+nc$ whenever m and n are integers
- What is the assumption? What is the conclusion?



Rules of Inference: Modus Ponens

- Intuitively, modus ponens (or law of detachment) can be described as the inference:

p implies q; p is true; therefore q holds

- In logic terminology, modus ponens is the tautology:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

- Note: ‘therefore’ is sometimes denoted \therefore , so we have:

$$p \rightarrow q \equiv p \therefore q$$

Rules of Inference: Addition

- Addition involves the tautology

$$p \rightarrow (p \vee q)$$

- Intuitively,
 - if we know that p is true
 - we can conclude that either p or q are true (or both)
- In other words: $p \therefore (p \vee q)$
- Example: I read the newspaper today,
therefore I read the newspaper or I ate custard
 - Note that these are not mutually exclusive

Rules of Inference: Simplification

- Simplification is based on the tautology

$$(p \wedge q) \rightarrow p$$

- So we have: $(p \wedge q) \therefore p$

- Example: Prove that if $0 < x < 10$, then $x \geq 0$

1. $0 < x < 10 \equiv (0 < x) \wedge (x < 10)$

2. $(x > 0) \wedge (x < 10) \rightarrow (x > 0)$

by simplification

3. $(x > 0) \rightarrow (x > 0) \vee (x = 0)$

by addition

4. $(x > 0) \vee (x = 0) \equiv (x \geq 0)$

Q.E.D.

QED= Latin word for “quod erat demonstrandum” meaning “that which was to be demonstrated” or “that which was to be shown”.

Rules of inference: Conjunction

- The conjunction is almost trivially intuitive. It is based on the following tautology:

$$((p) \wedge (q)) \rightarrow (p \wedge q)$$

- Note the subtle difference though:
 - On the left-hand side, we independently know p and q to be true
 - Therefore, we conclude, on the right-hand side, that a logical conjunction is true

Rules of Inference: Modus Tollens

- Similar to the modus ponens, modus tollens is based on the following tautology

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$$

- In other words:
 - If we know that q is not true
 - And that p implies q
 - Then we can conclude that p does not hold either
- Example
 - If you are New York State citizen student, then you pay in-state tuition fee at SUNY.
 - Tom pay out-state tuition fee at SUNY.
 - Therefore we can conclude that Tom is not a New York State citizen student.

Rules of Inference: Contrapositive

- The contrapositive is the following tautology

$$(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$$

- Usefulness
 - If you are having trouble proving the p implies q in a direct manner
 - You can try to prove the contrapositive instead!

Rules of Inference: Hypothetical Syllogism

- Hypothetical syllogism is based on the following tautology

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

- Essentially, this shows that the rules of inference are, in a sense, transitive
- Example:
 - If you don't get a job, you won't have money
 - If you don't have money, you will starve.
 - Therefore, if you don't get a job, you'll starve

Rules of Inference: Disjunctive Syllogism

- A disjunctive syllogism is formed on the basis of the tautology

$$((p \vee q) \wedge \neg p) \rightarrow q$$

- Reading this in English, we see that
 - If either p or q hold and we know that p does not hold
 - Then we can conclude that q must hold
- Example
 - The sky is either blue or grey
 - Well it isn't blue
 - Therefore, the sky is grey

Rules of Inference: Resolution

- For resolution, we have the following tautology

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$

- Essentially,
 - If we have two true disjunctions that have mutually exclusive propositions
 - Then we can conclude that the disjunction of the two non-mutually exclusive propositions is true

Proofs: Example (1)

Theorem: *The sum of two odd integers is even*

- Let n, m be two odd integers.
- Every odd integer x can be written as $x=2k+1$ for some integer k
- Therefore, let $n = 2k_1+1$ and $m=2k_2+1$

- Consider

$$n+m = (2k_1+1)+(2k_2+1)$$

$$= 2k_1+ 2k_2+1+1$$

Associativity/Commutativity

$$= 2k_1+ 2k_2+2$$

Algebra

$$= 2(k_1+ k_2+1)$$

Factoring

- By definition $2(k_1+k_2+1)$ is even, therefore $n+m$ is even *QED*

Proofs: Example (2)

- Assume that the statements below hold:
 - $(p \rightarrow q)$
 - $(r \rightarrow s)$
 - $(r \vee p)$
- Assume that q is false
- Show that s must be true

Proofs: Example (2)

1. $(p \rightarrow q)$
2. $(r \rightarrow s)$
3. $(r \vee p)$
4. $\neg q$
5. $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ by modus tollens on 1 + 4
6. $(r \vee p) \wedge \neg p \rightarrow r$ by disjunctive syllogism 3 + 6
7. $(r \wedge (r \rightarrow s)) \rightarrow s$ by modus ponens 2 + 6

If and Only If

- If you are asked to show an equivalence
$$p \leftrightarrow q \text{ “if and only if”}$$
- You must show an implication in both directions
- That is, you can show (independently or via the same technique) that $(p \rightarrow q)$ and $(q \rightarrow p)$
- Example
 - Show that x is odd iff x^2+2x+1 is even

Example (iff)

$$x \text{ is odd} \iff x=2k+1, k \in \mathbb{Z}$$

$$\iff x+1 = 2k+2$$

$$\iff x+1 = 2(k+1)$$

$$\iff x+1 \text{ is even}$$

$$\iff (x+1)^2 \text{ is even} \quad \text{Since } x \text{ is even iff } x^2 \text{ is even}$$

$$\iff x^2+2x+1 \text{ is even}$$

by definition

algebra

factoring

by definition

algebra

QED

Next class

- Topic: Proofs
- Pre-class reading: Chap 1.5-1.8

