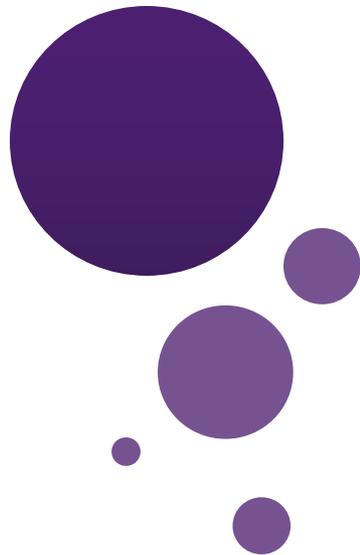




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Lecture 7: Set Operations And Introduction to Functions



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Recap Previous Lecture

- Definition of set, set builder (intension or extension).
- Null set, singleton set, subset, power set
- Set Equivalence
 - $A \subseteq B$ and $B \subseteq A$
 - $(\forall x (x \in A \Rightarrow x \in B)) \wedge (\forall x (x \in B \Rightarrow x \in A))$
- Cartesian Product
 - $A_1 \times A_2 \times \dots \times A_n = \{ (a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i=1, 2, \dots, n \}$

Outline

- Set Operations
- Set Equivalences
- Generalized Unions and Intersection
- Computer Representation of Sets
- Introduction to Functions

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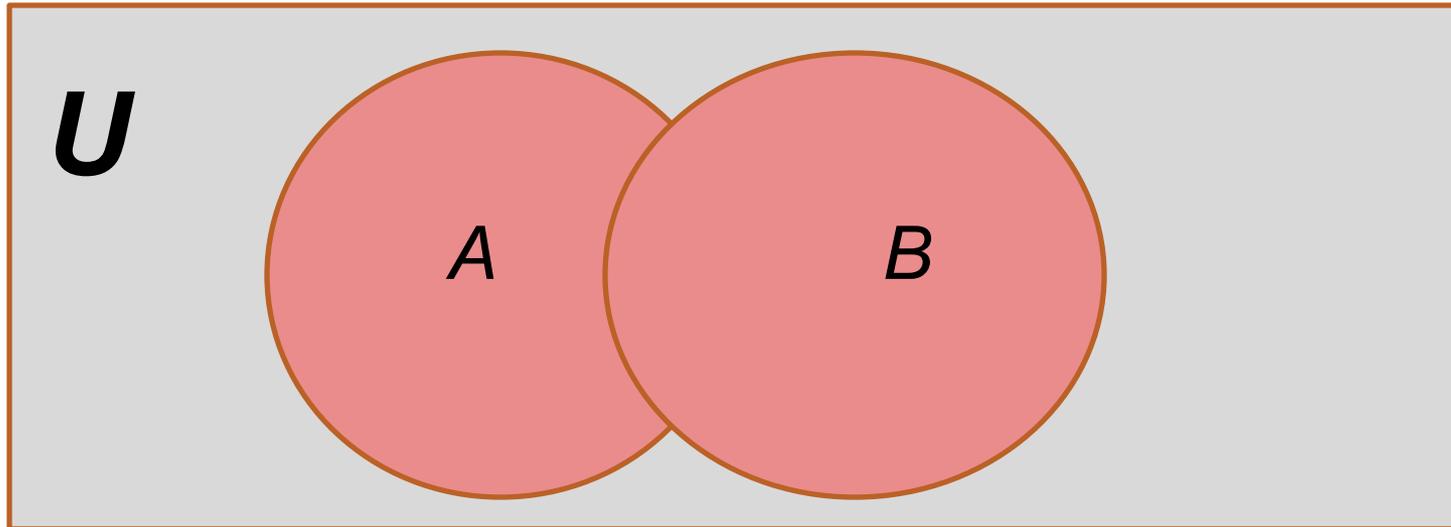
Set Operations

- Arithmetic operators (+, -, \times , \div) can be used on pairs of numbers to give us new numbers
- Similarly, set operators exist and act on two sets to give us new sets
 - Union \cup
 - Intersection \cap
 - Set difference \setminus
 - Set complement \overline{S}
 - Generalized union \bigcup
 - Generalized intersection \bigcap

Set Operators: Union

- **Definition:** The **union** of two sets A and B is the set that contains all elements in A , B , or both. We write:

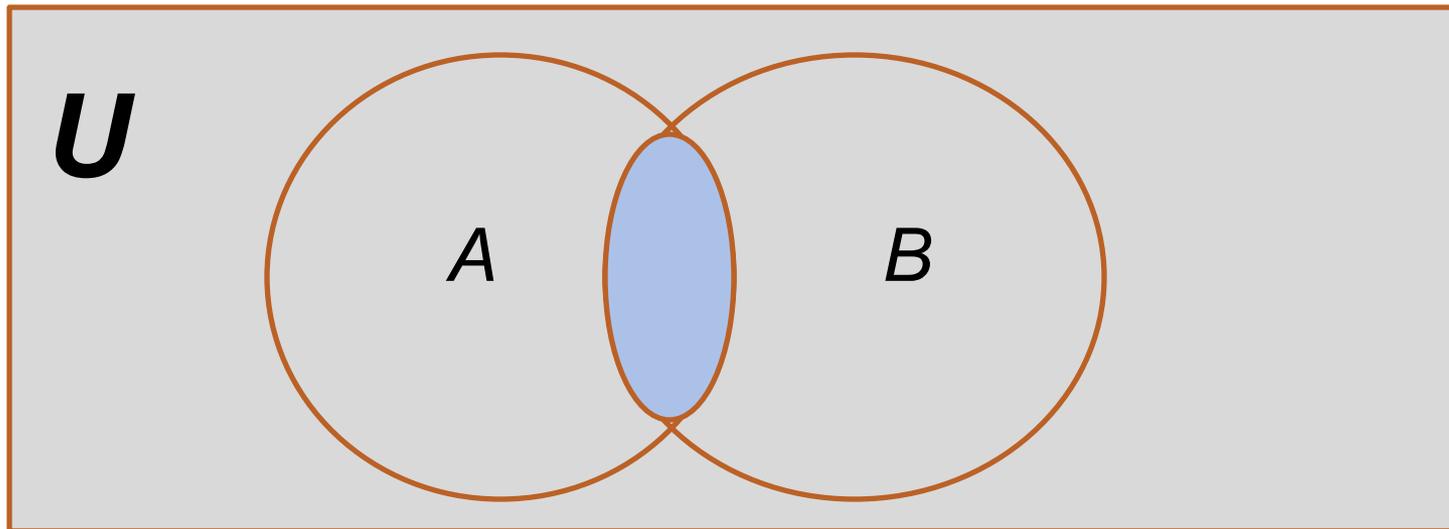
$$A \cup B = \{ x \mid (a \in A) \vee (b \in B) \}$$



Set Operators: Intersection

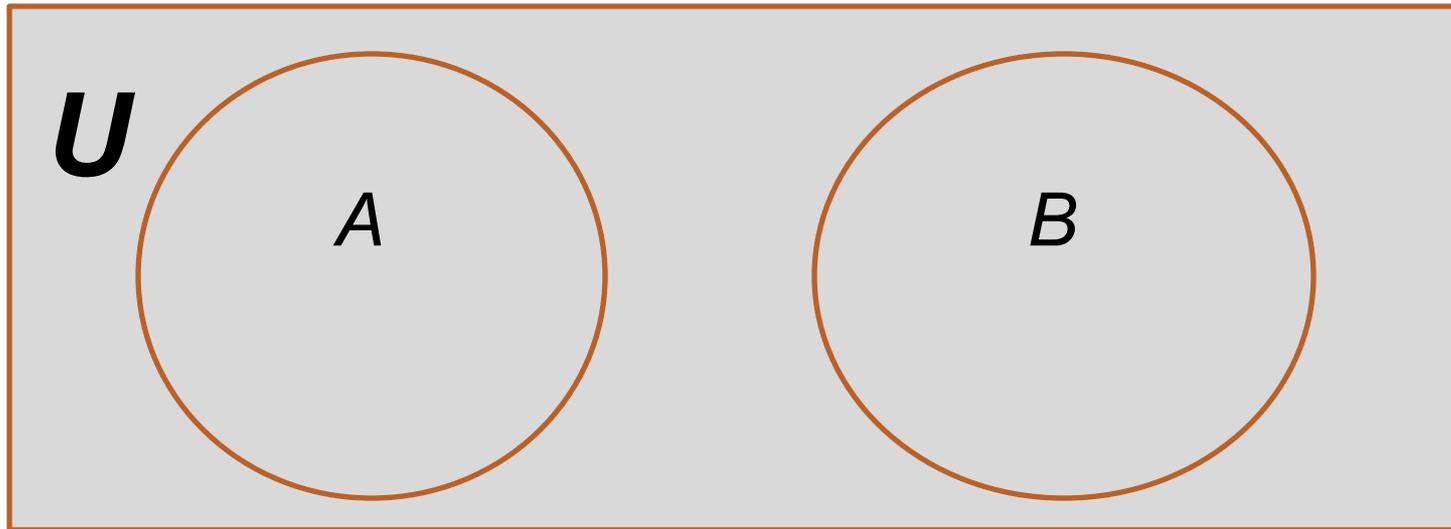
- **Definition:** The **intersection** of two sets A and B is the set that contains all elements that are element of both A and B . We write:

$$A \cap B = \{ x \mid (a \in A) \wedge (b \in B) \}$$



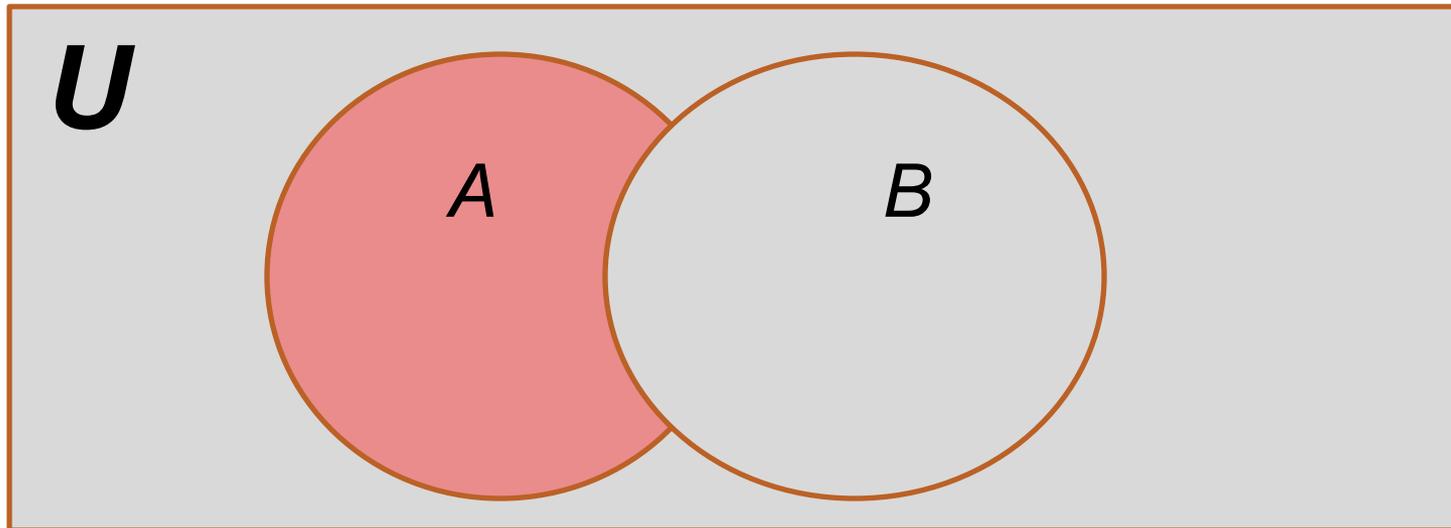
Disjoint Sets

- **Definition:** Two sets are said to be **disjoint** if their intersection is the empty set: $A \cap B = \emptyset$



Set Difference

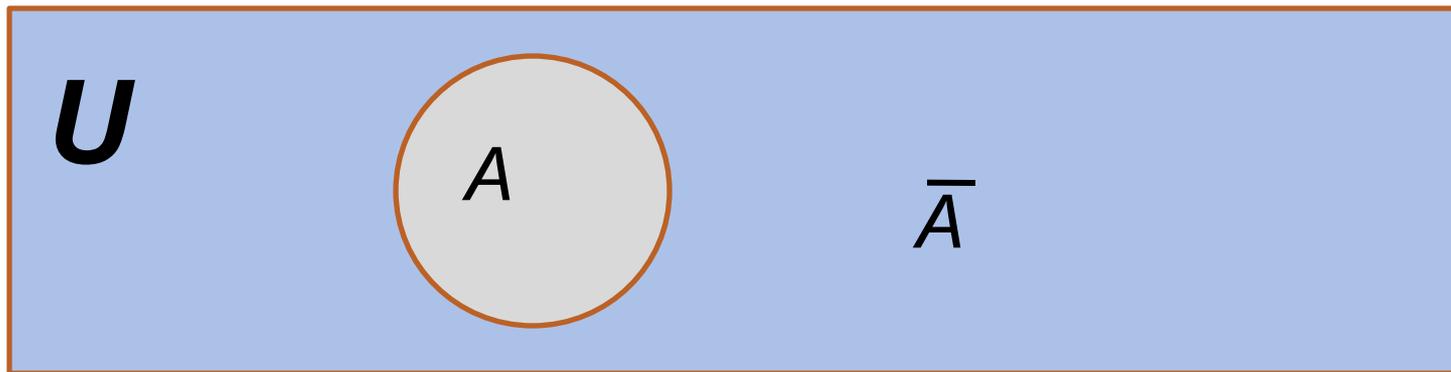
- **Definition:** The **difference** of two sets A and B , denoted $A \setminus B$ (\setminus setminus) or $A - B$, is the set containing those elements that are in A but not in B



Set Complement

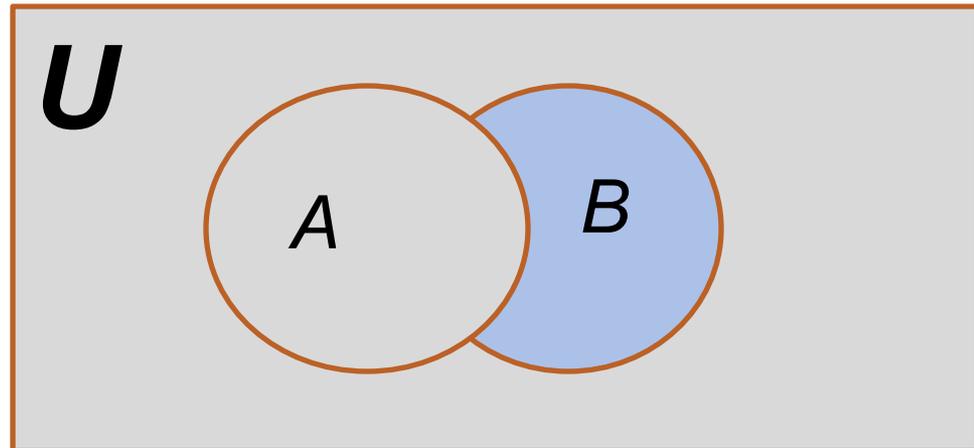
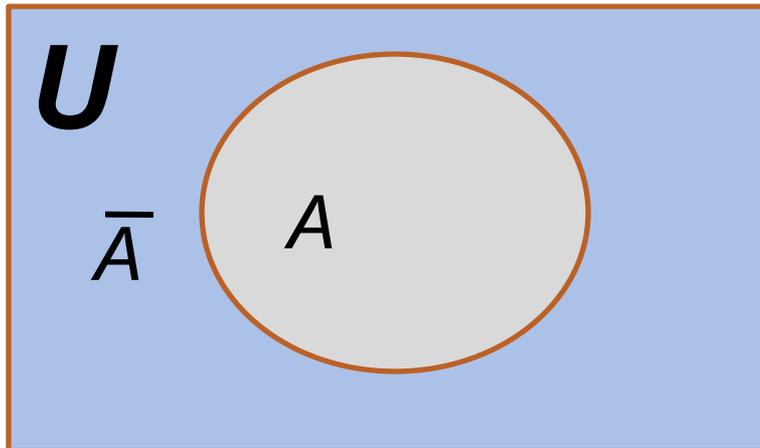
- **Definition:** The **complement** of a set A , denoted \bar{A} ($\bar{}$), consists of all elements not in A . That is the difference of the universal set and U : $U \setminus A$

$$\bar{A} = A^c = \{x \mid x \notin A\}$$



Set Complement: Absolute & Relative

- Given the Universe U , and $A, B \subset U$.
- The (absolute) complement of A is $A^c = U \setminus A$
- The (relative) complement of A in B is $B \setminus A$



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Proving Set Equivalences

- Recall that to prove such identity, we must show that:
 1. The left-hand side is a subset of the right-hand side
 2. The right-hand side is a subset of the left-hand side
 3. Then conclude that the two sides are thus equal
- The book proves several of the standard set identities
- We will give a couple of different examples here

Proving Set Equivalences: Example A (1)

- Let
 - $A = \{x \mid x \text{ is even}\}$
 - $B = \{x \mid x \text{ is a multiple of } 3\}$
 - $C = \{x \mid x \text{ is a multiple of } 6\}$
- Show that $A \cap B = C$

Proving Set Equivalences: Example A (2)

- **$A \cap B \subseteq C$** : $\forall x \in A \cap B$
 - \Rightarrow x is a multiple of 2 and x is a multiple of 3
 - \Rightarrow we can write $x=2 \times 3 \times k$ for some integer k
 - $\Rightarrow x=6k$ for some integer $k \Rightarrow x$ is a multiple of 6
 - $\Rightarrow x \in C$
- **$C \subseteq A \cap B$** : $\forall x \in C$
 - $\Rightarrow x$ is a multiple of 6 $\Rightarrow x=6k$ for some integer k
 - $\Rightarrow x=2(3k)=3(2k) \Rightarrow x$ is a multiple of 2 and of 3
 - $\Rightarrow x \in A \cap B$

Proving Set Equivalences: Example B (1)

- An alternative prove is to use **membership tables** where an entry is
 - 1 if a chosen (but fixed) element is in the set
 - 0 otherwise
- Example: Show that

$$\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$$

Proving Set Equivalences: Example B (2)

| A | B | C | $A \cap B \cap C$ | $\overline{A \cap B \cap C}$ | \overline{A} | \overline{B} | \overline{C} | $\overline{A \cup B \cup C}$ |
|---|---|---|-------------------|------------------------------|----------------|----------------|----------------|------------------------------|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |

- 1 under a set indicates that an element is in the set
- If the columns are equivalent, we can conclude that indeed the two sets are equal

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Generalizing Set Operations: Union and Intersection

- In the previous example, we showed De Morgan's Law generalized to unions involving 3 sets
- In fact, De Morgan's Laws hold for any finite set of sets
- Moreover, we can generalize set operations union and intersection in a straightforward manner to any finite number of sets

Generalized Union

- **Definition:** The **union of a collection of sets** is the set that contains those elements that are members of at least one set in the collection

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

LaTeX: $\$\Bigcup_{i=1}^n A_i=A_1\cup A_2 \cup \dots \cup A_n\$$

Generalized Intersection

- **Definition:** The **intersection of a collection of sets** is the set that contains those elements that are members of every set in the collection

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

LaTeX: $\$\Bigcap_{i=1}^n A_i=A_1\cap A_2 \cap\ldots\cap A_n\$$

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Computer Representation of Sets (1)

- There really aren't ways to represent infinite sets by a computer since a computer has a finite amount of memory
- If we assume that the universal set U is finite, then we can easily and effectively represent sets by bit vectors
- Specifically, we force an ordering on the objects, say:

$$U = \{a_1, a_2, \dots, a_n\}$$

- For a set $A \subseteq U$, a bit vector can be defined as, for $i=1, 2, \dots, n$
 - $b_i=0$ if $a_i \notin A$
 - $b_i=1$ if $a_i \in A$

Computer Representation of Sets (2)

- Examples
 - Let $U=\{0,1,2,3,4,5,6,7\}$ and $A=\{0,1,6,7\}$
 - The bit vector representing A is: 1100 0011
 - How is the empty set represented?
 - How is U represented?
- Set operations become trivial when sets are represented by bit vectors
 - Union is obtained by making the bit-wise OR
 - Intersection is obtained by making the bit-wise AND

Computer Representation of Sets (3)

- Let $U=\{0,1,2,3,4,5,6,7\}$, $A=\{0,1,6,7\}$,
 $B=\{0,4,5\}$
- What is the bit-vector representation of B ?
- Compute, bit-wise, the bit-vector representation of $A \cap B$
- Compute, bit-wise, the bit-vector representation of $A \cup B$
- What sets do these bit vectors represent?

Programming Question

- Using bit vector, we can represent sets of cardinality equal to the size of the vector
- What if we want to represent an arbitrary sized set in a computer (i.e., that we do not know a priori the size of the set)?
- What data structure could we use?

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Introduction

- You have already encountered function
 - $f(x,y) = x+y$
 - $f(x) = x$
 - $f(x) = \sin(x)$
- Here we will study functions defined on discrete **domains** and **ranges**
- We will generalize functions to mappings
- We may not always be able to write function in a 'neat way' as above

Definition: Function

- **Definition:** A function f from a set A to a set B is an assignment of **exactly one** element of B to **each** element of A .
- We write $f(a)=b$ if b is the unique element of B assigned by the function f to the element $a \in A$.
- If f is a function from A to B , we write

$$f: A \rightarrow B$$

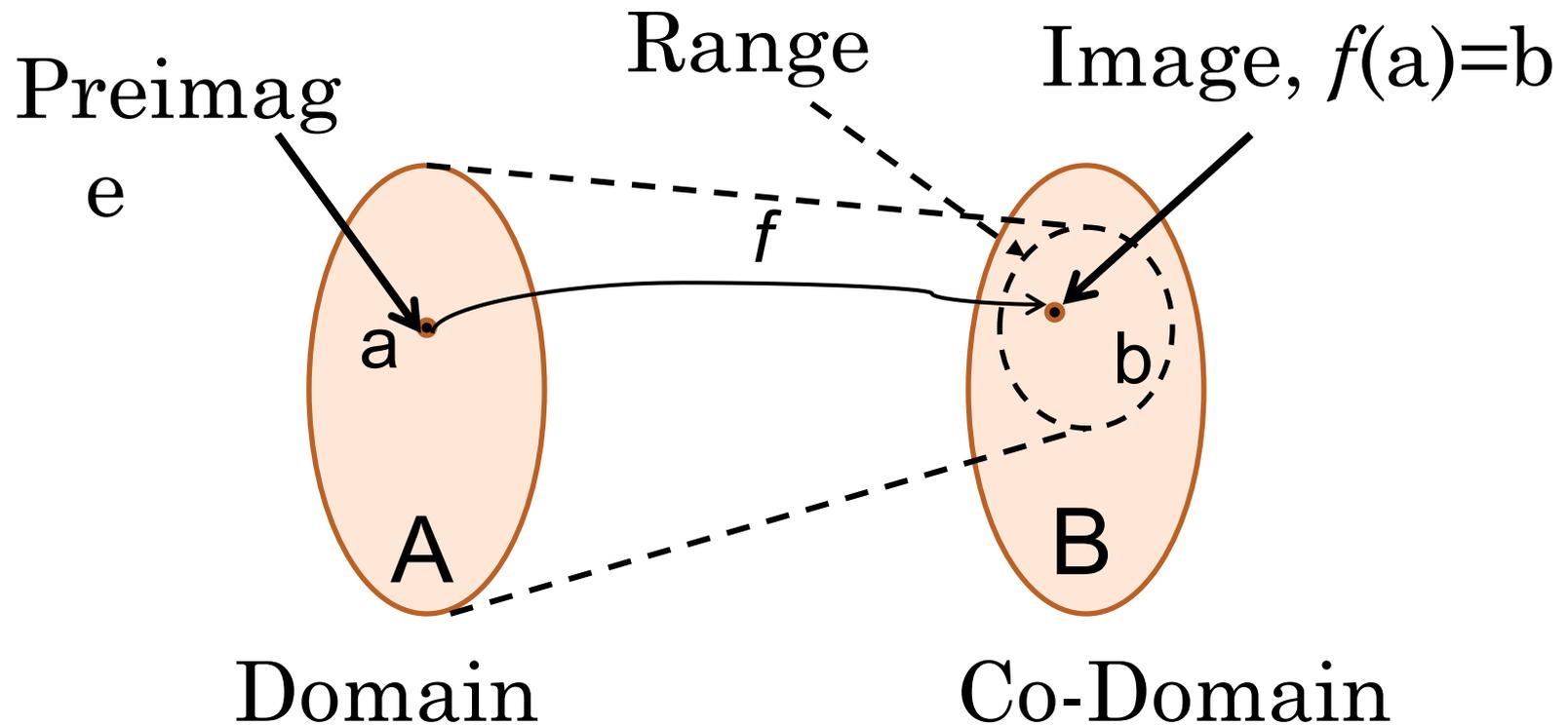
This can be read as ‘ f maps A to B ’

- Note the subtlety
 - Each and every element of A has a single mapping
 - Each element of B may be mapped to by several elements in A or not at all

Terminology

- Let $f: A \rightarrow B$ and $f(a)=b$. Then we use the following terminology:
 - A is the domain of f , denoted $\text{dom}(f)$
 - B is the co-domain of f
 - b is the image of a
 - a is the preimage (antecedent) of b
 - The range of f is the set of all images of elements of A, denoted $\text{rng}(f)$

Function: Visualization



A function, $f: A \rightarrow B$

More Definitions (1)

- **Definition:** Let f_1 and f_2 be two functions from a set A to \mathbb{R} . Then f_1+f_2 and f_1f_2 are also function from A to \mathbb{R} defined by:
 - $(f_1+f_2)(x) = f_1(x) + f_2(x)$
 - $f_1f_2(x) = f_1(x)f_2(x)$
- **Example:** Let $f_1(x)=x^4+2x^2+1$ and $f_2(x)=2-x^2$
 - $(f_1+f_2)(x) = x^4+2x^2+1+2-x^2 = x^4+x^2+3$
 - $f_1f_2(x) = (x^4+2x^2+1)(2-x^2) = -x^6+3x^2+2$

More Definitions (2)

- **Definition:** Let $f: A \rightarrow B$ and $S \subseteq A$. The **image of the set S** is the subset of B that consists of all the images of the elements of S . We denote the image of S by $f(S)$, so that

$$f(S) = \{ f(s) \mid \forall s \in S \}$$

- Note there that the image of S is a set and not an element.

Image of a set: Example

- Let:
 - $A = \{a_1, a_2, a_3, a_4, a_5\}$
 - $B = \{b_1, b_2, b_3, b_4, b_5\}$
 - $f = \{(a_1, b_2), (a_2, b_3), (a_3, b_3), (a_4, b_1), (a_5, b_4)\}$
 - $S = \{a_1, a_3\}$
- Draw a diagram for f
- What is the:
 - Domain, co-domain, range of f ?
 - Image of S , $f(S)$?

More Definitions (3)

- **Definition:** A function f whose domain and codomain are subsets of the set of real numbers (\mathbb{R}) is called
 - **strictly increasing** if $f(x) < f(y)$ whenever $x < y$ and x and y are in the domain of f .
 - **strictly decreasing** if $f(x) > f(y)$ whenever $x < y$ and x and y are in the domain of f .
- A function that is increasing or decreasing is said to be **monotonic**

Next class

- Topic: Functions
- Pre-class reading: Chap 2.3

