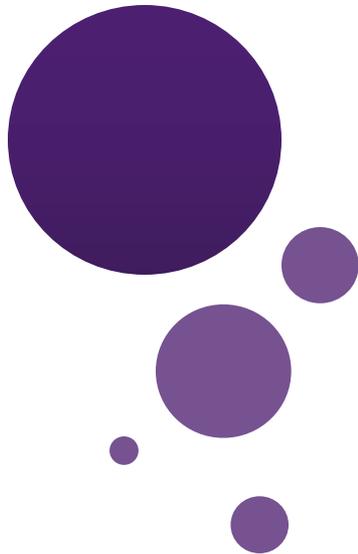UNIVERSITY
AT ALBANY

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Lecture 8: Functions

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Recap Previous Lecture

- ## Set operators on two sets:
  - Union $\cup$ and Intersection $\cap$
  - Set difference $\setminus$
  - Set complement $\overline{S}$
  - Generalized union $\bigcup$
  - Generalized intersection $\bigcap$
- ## Computer Representation of Sets: bit vector
- ## Definition of function:
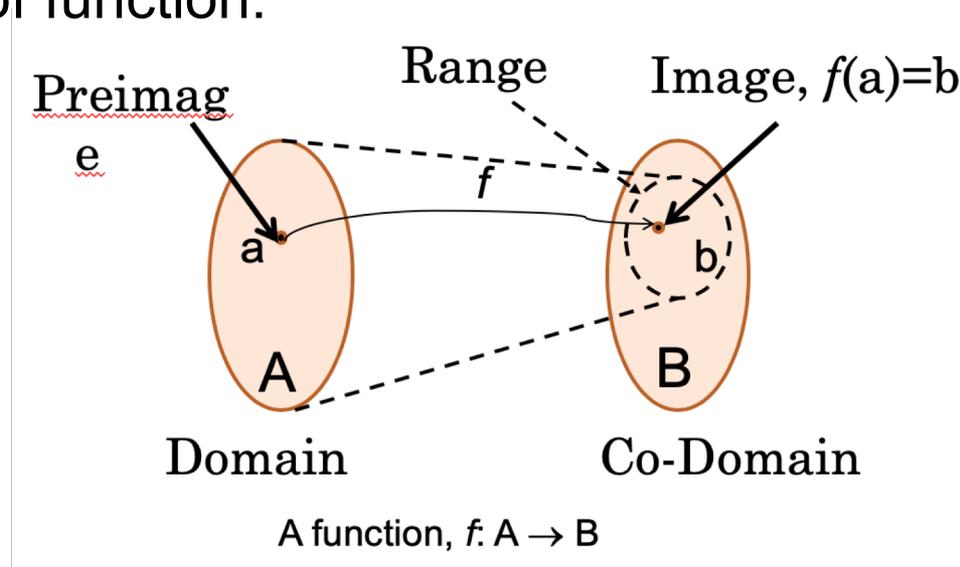
A function, $f\colon A \to B$

Outline

- Properties

- Inverse functions

- Operators

- Important functions

Outline

- **Properties**
- Inverse functions
- Operators
- Important functions

Definition: Injection

- **Definition**: A function *f* is said to be <u>one-to-one</u> or <u>injective</u> (or an injection) if

  $\forall$ x and y in in the domain of *f*, *f*(x)=*f*(y) $\Rightarrow$ x=y

- Intuitively, an injection simply means that each element in the range has at most one preimage (antecedent)

- It is useful to think of the contrapositive of this definition

  $$x \neq y \implies f(x) \neq f(y)$$

Definition: Surjection

- **Definition**: A function f: A$\rightarrow$B is called <u>onto</u> or <u>surjective</u> (or an surjection) if

$$\forall \, b \in B, \exists \, a \in A \text{ with } f(a)=b$$

- Intuitively, a surjection means that every element in the codomain is mapped into (i.e., it is an image, has an antecedent)

- Thus, the range is the same as the codomain

Definition: Bijection

- **Definition**: A function *f* is a <u>one-to-one</u> correspondence (or a <u>bijection</u>), if is both one-to-one (injective) and onto (surjective)

- One-to-one correspondences are important because they endow a function with an <u>inverse</u>.

- They also allow us to have a concept cardinality for infinite sets

- Let's look at a few examples to develop a feel for these definitions…

Functions: Example 1
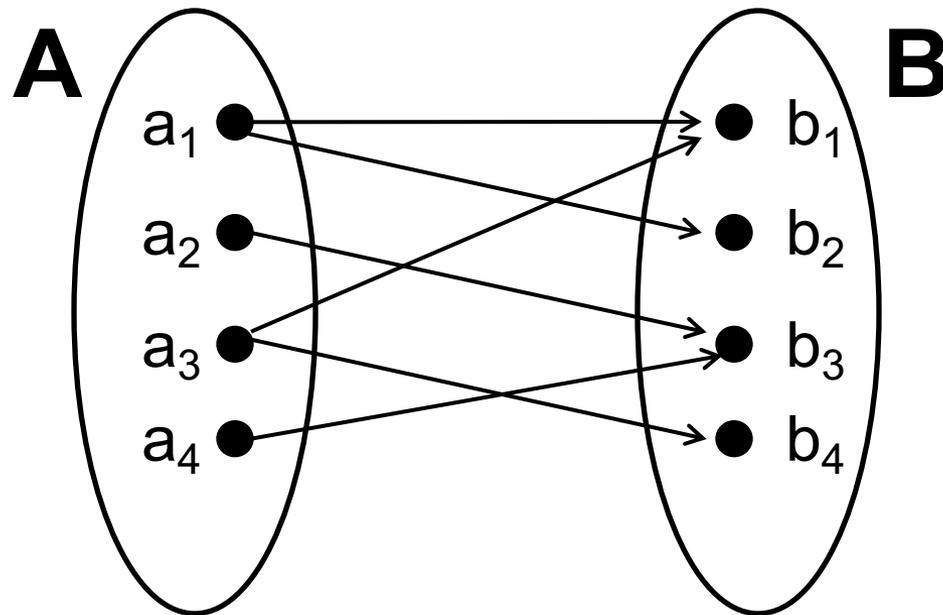
- Is this a function? Why?

- No, because each of a_1, a_2 has two images

Functions: Example 2
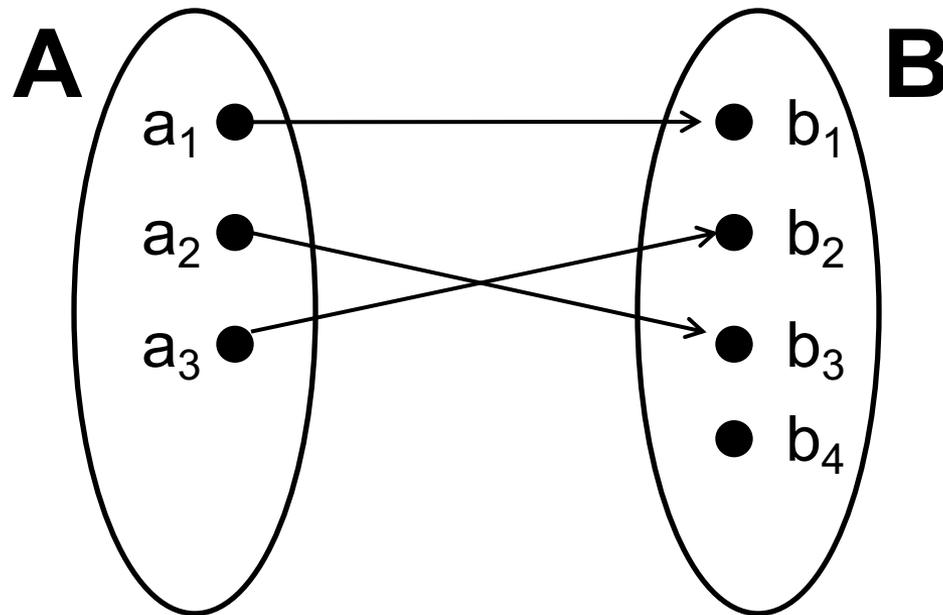

A ... **B**

$a_1 \rightarrow b_1$
$a_2 \rightarrow b_1$
$a_3 \rightarrow b_2$
$a_4 \rightarrow b_3$
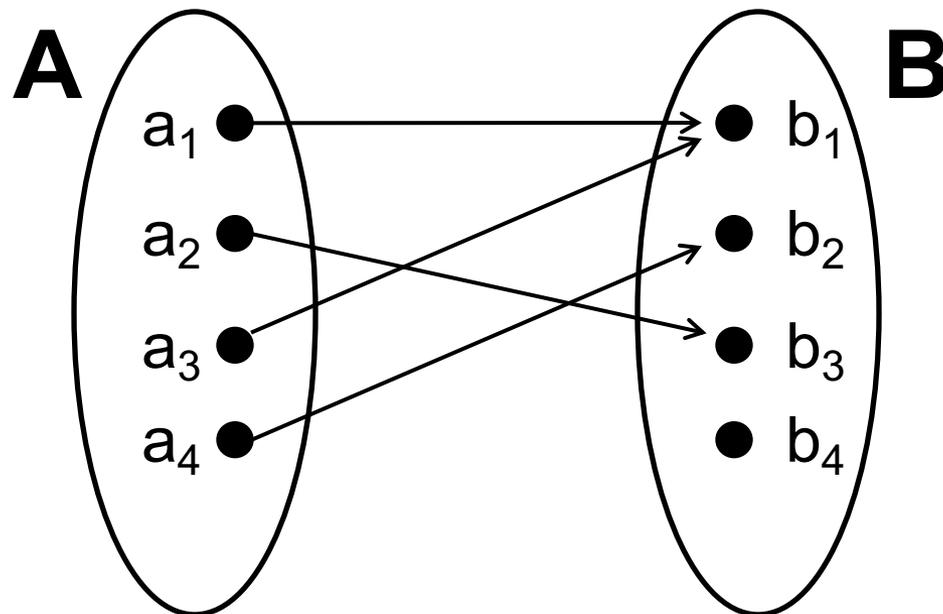
- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

No, b_1 has 2 preimages

No, b_4 has no preimage
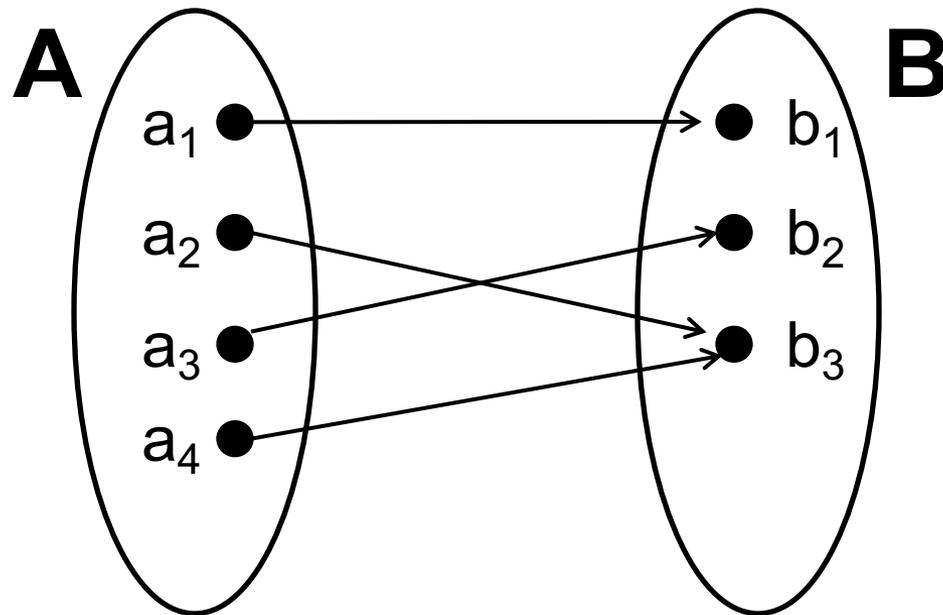
Functions: Example 3

- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

Yes, no b_i has 2 preimages

No, b_4 has no preimage

Functions: Example 4

A

a_1
a_2
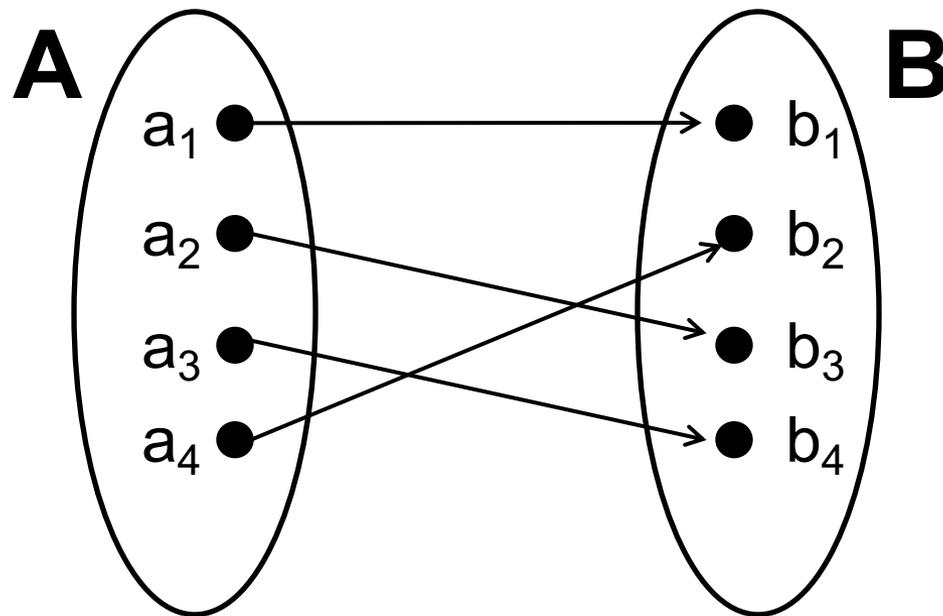a_3
a_4

B

b_1
b_2
b_3

- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

No, b_3 has 2 preimages

Yes, every b_i has a preimage

Functions: Example 5

A

a_1
a_2
a_3
a_4

B

b_1
b_2
b_3
b_4

- Is this a function
  - One-to-one (injective)?
  - Onto (surjective)?

Thus, it is a bijection or a one-to-one correspondence

Exercice 1

- Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by
$$f(x) = 2x - 3$$
- What is the domain, codomain, range of f?
- Is f one-to-one (injective)?
- Is f onto (surjective)?
- Clearly, dom(f)=$\mathbb{Z}$. To see what the range is, note that:

$b \in$ rng(f) $\Leftrightarrow$ b=2a-3, with a$\in \mathbb{Z}$

$\Leftrightarrow$ b=2(a-2)+1

$\Leftrightarrow$ b is odd

Exercise 1 (cont'd)

- Thus, the range is the set of all odd integers

- Since the range and the codomain are different (i.e., $rng(f) \neq \mathbb{Z}$), we can conclude that f is not onto (surjective)

- However, f is one-to-one injective. Using simple algebra, we have:

$$f(x_1) = f(x_2) \Rightarrow 2x_1\text{-}3 = 2x_2\text{-}3 \Rightarrow x_1 = x_2 \quad \text{QED}$$

Exercise 2

- Let *f* be as before

$$f(x)=2x-3$$

   but now we define $f:\mathbb{N}\to\mathbb{N}$

- What is the domain and range of *f*?
- Is *f* onto (surjective)?
- Is *f* one-to-one (injective)?

- By changing the domain and codomain of *f*, *f* is not even a function anymore. Indeed, $f(1)=2\cdot 1-3=-1\notin\mathbb{N}$

Exercice 3

- Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = x^2 - 5x + 5$$

- Is this function
  - One-to-one?
  - Onto?

Exercice 3: Answer

- ## It is not one-to-one (injective)

  $f(x_1)=f(x_2) \Rightarrow x_1^2-5x_1+5=x_2^2-5x_2+5 \Rightarrow x_1^2-5x_1 = x_2^2-5x_2$

  $\Rightarrow x_1^2-x_2^2 = 5x_1-5x_2 \Rightarrow (x_1-x_2)(x_1+x_2) = 5(x_1-x_2)$

  $\Rightarrow (x_1+x_2) = 5$

  Many $x_1, x_2 \in \mathbb{Z}$ satisfy this equality. There are thus an infinite number of solutions. In particular, $f(2)=f(3)=-1$

- ## It is also not onto (surjective).

  The function is a parabola with a global minimum at (5/2,-5/4). Therefore, the function fails to map to any integer less than -1

- ## What would happen if we changed the domain/codomain?

Exercice 4

- Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = 2x^2 + 7x$$

- Is this function
  - One-to-one (injective)?
  - Onto (surjective)?
- Again, this is a parabola, it cannot be onto (where is the global minimum?)

Exercice 4: Answer

- However, it is one-to-one! Indeed:

$f(x_1)=f(x_2) \Rightarrow 2x_1^2+7x_1=2x_2^2 + 7x_2 \Rightarrow 2x_1^2 - 2x_2^2 = 7x_2 - 7x_1$

$\Rightarrow 2(x_1 - x_2)(x_1 + x_2) = 7(x_2 - x_1) \Rightarrow 2(x_1 + x_2) = -7 \Rightarrow (x_1 + x_2) = -7$

$\Rightarrow (x_1 + x_2) = -7/2$

But $-7/2 \notin Z$. Therefore it must be the case that $x_1 = x_2$.

It follows that f is a one-to-one function.
QED

Exercise 5

- Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by

$$f(x) = 3x^3 - x$$

- Is this function
  - One-to-one (injective)?
  - Onto (surjective)?

Exercice 5: *f* is one-to-one

- To check if *f* is one-to-one, again we suppose that for $x_1, x_2 \in \mathbb{Z}$ we have $f(x_1) = f(x_2)$

  $f(x_1) = f(x_2) \Rightarrow 3x_1^3 - x_1 = 3x_2^3 - x_2$

  $\Rightarrow 3x_1^3 - 3x_2^3 = x_1 - x_2$

  $\Rightarrow 3(x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2) = (x_1 - x_2)$

  $\Rightarrow (x_1^2 + x_1 x_2 + x_2^2) = 1/3$

  which is impossible because $x_1, x_2 \in \mathbb{Z}$

  thus, *f* is one-to-one

Exercice 5: *f* is <u>not</u> onto

- Consider the counter example *f*(a)=1
- If this were true, we would have

  $3a^3 - a = 1 \Rightarrow a(3a^2 - 1) = 1$ where a and $(3a^2 - 1) \in \mathbb{Z}$

- The only time we can have the product of two integers equal to 1 is when they are both equal to 1 or -1

- Neither 1 nor -1 satisfy the above equality

  - Thus, we have identified $1 \in \mathbb{Z}$ that does not have an antecedent and *f* is not onto (surjective)

Outline

- Properties
- **Inverse functions (examples)**
- Operators
- Important functions

Inverse Functions (1)

- **Definition**: Let f: A$\rightarrow$B be a bijection. The <u>inverse</u> function of f is the function that assigns to an element b$\in$B the unique element a$\in$A such that f(a)=b

- The inverse function is denote f^1.

- When f is a bijection, its inverse exists and

$$f(a)=b \iff f^1(b)=a$$

Inverse Functions (2)

- Note that by definition, a function can have an inverse if and only if it is a bijection. Thus, we say that a bijection is <u>invertible</u>

- Why must a function be bijective to have an inverse?
  - Consider the case where f is not one-to-one (not injective). This means that some element b∈B has more than one antecedent in A, say a_1 and a_2. How can we define an inverse? Does $f^1(b)=a_1$ or a_2?
  - Consider the case where f is not onto (not surjective). This means that there is some element b∈B that does not have any preimage a∈A. What is then $f^1(b)$?

Inverse Functions: Representation

f(a)

a

f^{-1}(b)

b

A

B

Domain

Co-Domain
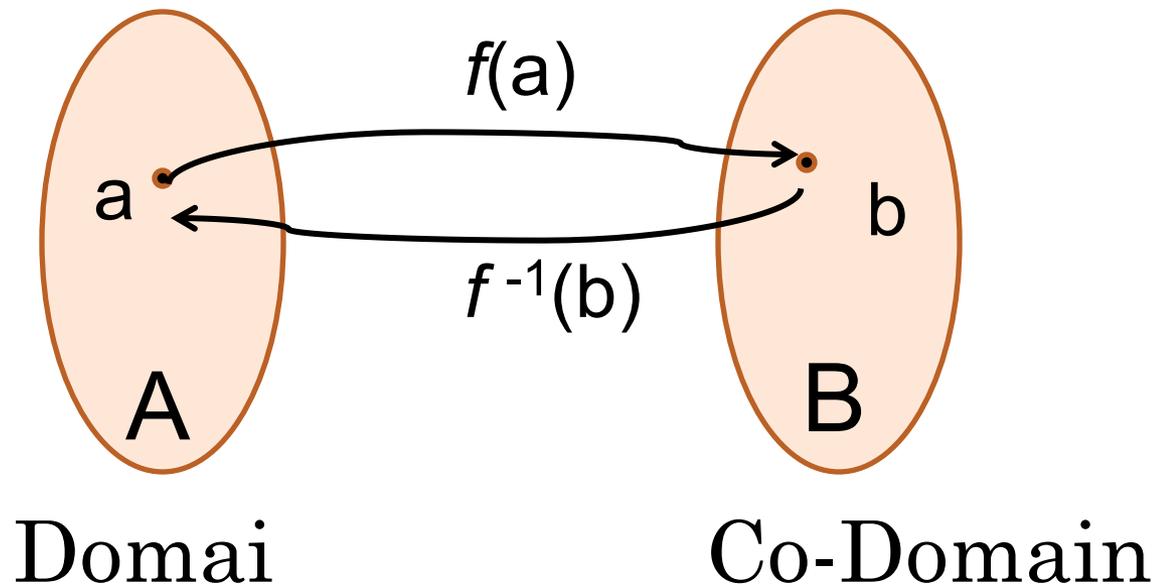
A function and its inverse

Inverse Functions: Example 1

- Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 2x - 3$$

- What is f^{-1}?

  1. We must verify that f is invertible, that is, is a bijection.   We prove that is one-to-one (injective) and onto (surjective). It is.

  2. To find the inverse, we use the substitution

     - Let $f^{-1}(y) = x$
     - And y=2x-3, which we solve for x. Clearly, x= (y+3)/2
     - So, $f^{-1}(y) = (y+3)/2$

Inverse Functions: Example 2

- Let $f(x)=x^2$. What is f^{-1}?
- No domain/codomain has been specified.
- Say $f: \mathbb{R} \rightarrow \mathbb{R}$
  - Is f a bijection? Does its inverse exist?
    – Answer: No
- Say we specify that f: A $\rightarrow$ B where

  $$A=\{x \in \mathbb{R} \mid x \leq 0\} \text{ and } B=\{y \in \mathbb{R} \mid y \geq 0\}$$

    – Is f a bijection? Does its inverse exist?
    – Answer: Yes, the function <u>becomes</u> a bijection and thus, has an inverse

Inverse Functions: Example 2 (cont')

- To find the inverse, we let
  - $f^{-1}(y)=x$
  - $y=x^2$, which we solve for x
- Solving for x, we get $x=\pm\sqrt{y}$, but which one is it?
- Since dom(f) is all nonpositive and rng(f) is nonnegative, thus x must be nonpositive and
$$f^{-1}(y)= -\sqrt{y}$$
- From this, we see that <u>the domains/codomains are just as important to a function as the definition of the function </u>itself

Inverse Functions: Example 3

- Let $f(x)=2^x$
  - What should the domain/codomain be for this function to be a bijection?
  - What is the inverse?
- The function should be $f: \mathbb{R} \rightarrow \mathbb{R}^+$
- Let $f^{-1}(y)=x$ and $y=2^x$, solving for x we get $x=\log_2(y)$. Thus, $f^{-1}(y)=\log_2(y)$
- What happens when we include 0 in the codomain?
- What happens when restrict either sets to $\mathbb{Z}$?

Outline

- Properties
- Inverse functions (examples)
- **Operators**
- Important functions

Function Composition (1)

- The value of functions can be used as the input to other functions

- **Definition**: Let $g{:}A{\rightarrow}B$ and $f{:}B \rightarrow C$. The <u>composition</u> of the functions f and g is

$$(f \circ g)\ (x) = f(g(x))$$

- $f{\circ}g$ is read as 'f circle g', or 'f composed with g', 'f following g', or just 'f of g'
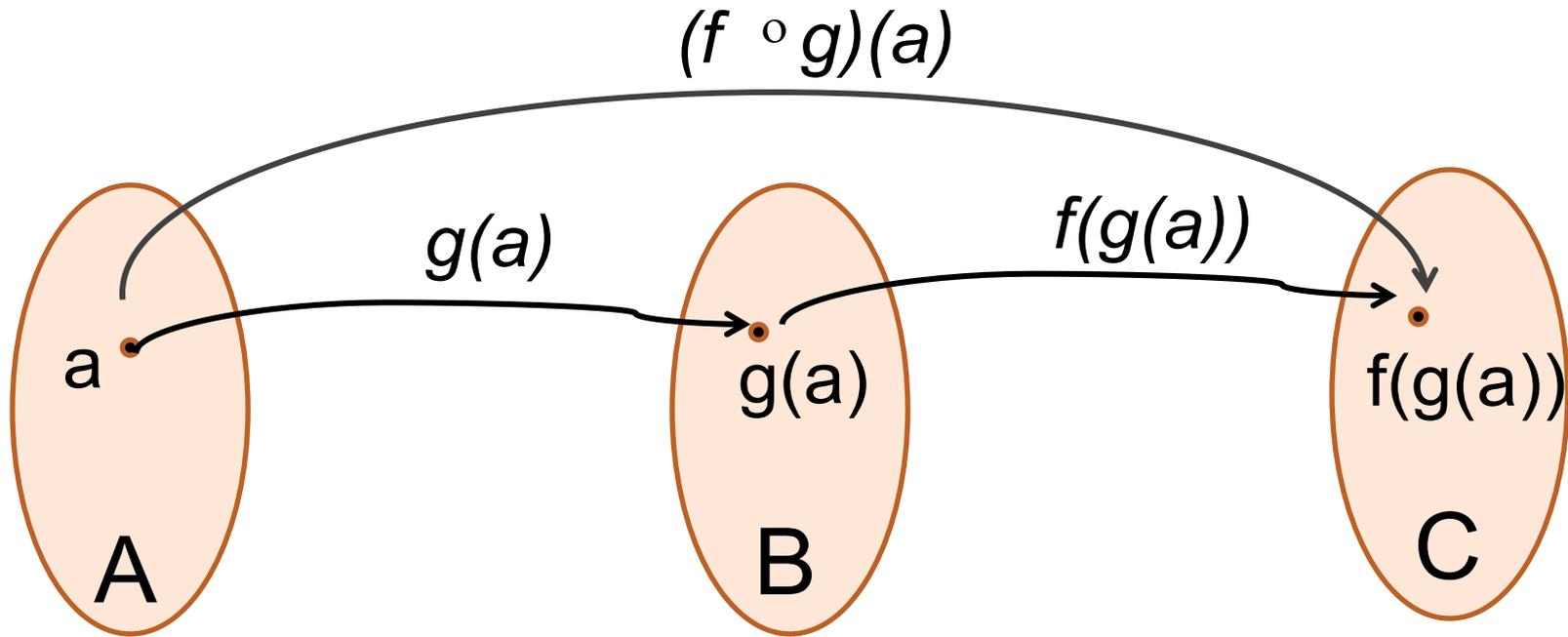
- In LaTeX: $\circ$

Function Composition (2)

- Because $(f \circ g)(x) = f(g(x))$, the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f

$$f \circ g \text{ is defined} \iff \text{rng}(g) \subseteq \text{dom}(f)$$

- The <u>order</u> in which you apply a function matters: you go from the inner most to the outer most

- It follows that $f \circ g$ is in general <u>not</u> the same as $g \circ f$

Composition: Graphical Representation

The composition of two functions

Composition: Example 1

- Let *f, g* be two functions on $\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 2x - 3$$

$$g(x) = x^2 + 1$$

- What are $f \circ g$ and $g \circ f$?
- We note that
  - *f* is bijective, thus dom(*f*)=rng(*f*)= codomain(*f*)= $\mathbb{R}$
  - For *g*, dom(*g*)= $\mathbb{R}$ but rng(*g*)={x$\in \mathbb{R}$ | x$\geq$1} $\subseteq \mathbb{R}^+$
  - Since rng(*g*)={x$\in \mathbb{R}$ | x$\geq$1} $\subseteq \mathbb{R}^+ \subseteq$ dom(*f*) $= \mathbb{R}$, $f \circ g$ is defined
  - Since rng(*f*)= $\mathbb{R} \subseteq$ dom(*g*) $= \mathbb{R}$, $g \circ f$ is defined

Composition: Example 1 (cont')

- Given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$

- $(f \circ g)(x) = f(g(x)) = f(x^2+1) = 2(x^2+1)-3$
  $$= 2x^2 - 1$$

- $(g \circ f)(x) = g(f(x)) = g(2x-3) = (2x-3)^2 +1$
  $$= 4x^2 - 12x + 10$$

Function Equality

- Although it is intuitive, we formally define what it means for two functions to be equal

- **Lemma**: Two functions *f* and *g* are <u>equal</u> if and only
  - dom(*f*) = dom(*g*)
  - $\forall$ a $\in$ dom(*f*) (*f*(a) = *g*(a))

Associativity

- The composition of function is not commutative ($f \circ g \neq g \circ f$), it is associative
- **Lemma**: The composition of functions is an associative operation, that is

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Outline

- Properties
- Inverse functions (examples)
- Operators
- **Important functions**

Important Functions: Identity

- **Definition**: The <u>identity</u> function on a set A is the function

$$\iota: A \to A$$

ι

defined by $\iota(a) = a$ for all $a \in A$.

- One can view the identity function as a composition of a function and its inverse:

$$\iota(a) = (f \circ f^1)(a) = (f^1 \circ f)(a)$$

- Moreover, the composition of any function *f* with the identity function is itself *f*:

$$(f \circ \iota)(a) = (\iota \circ f)(a) = f(a)$$

Inverses and Identity

- The identity function, along with the composition operation, gives us another characterization of <u>inverses</u> when a function has an inverse

- **Theorem**: The functions f: A$\rightarrow$B and g: B$\rightarrow$A are <u>inverses</u> if and only if

$$(g \circ f) = \iota_A \text{ and } (f \circ g) = \iota_B$$

where the ι_A and ι_B are the identity functions on sets A and B. That is,

$$\forall a \in A, b \in B \ (\ (g(f(a)) = a) \wedge (f(g(b)) = b) \ )$$

Important Functions: Absolute Value

- **Definition**: The <u>absolute value</u> function, denoted $|x|$, f $f:\mathbb{R} \rightarrow \{y \in \mathbb{R} \mid y \geq 0\}$. Its value is defined by

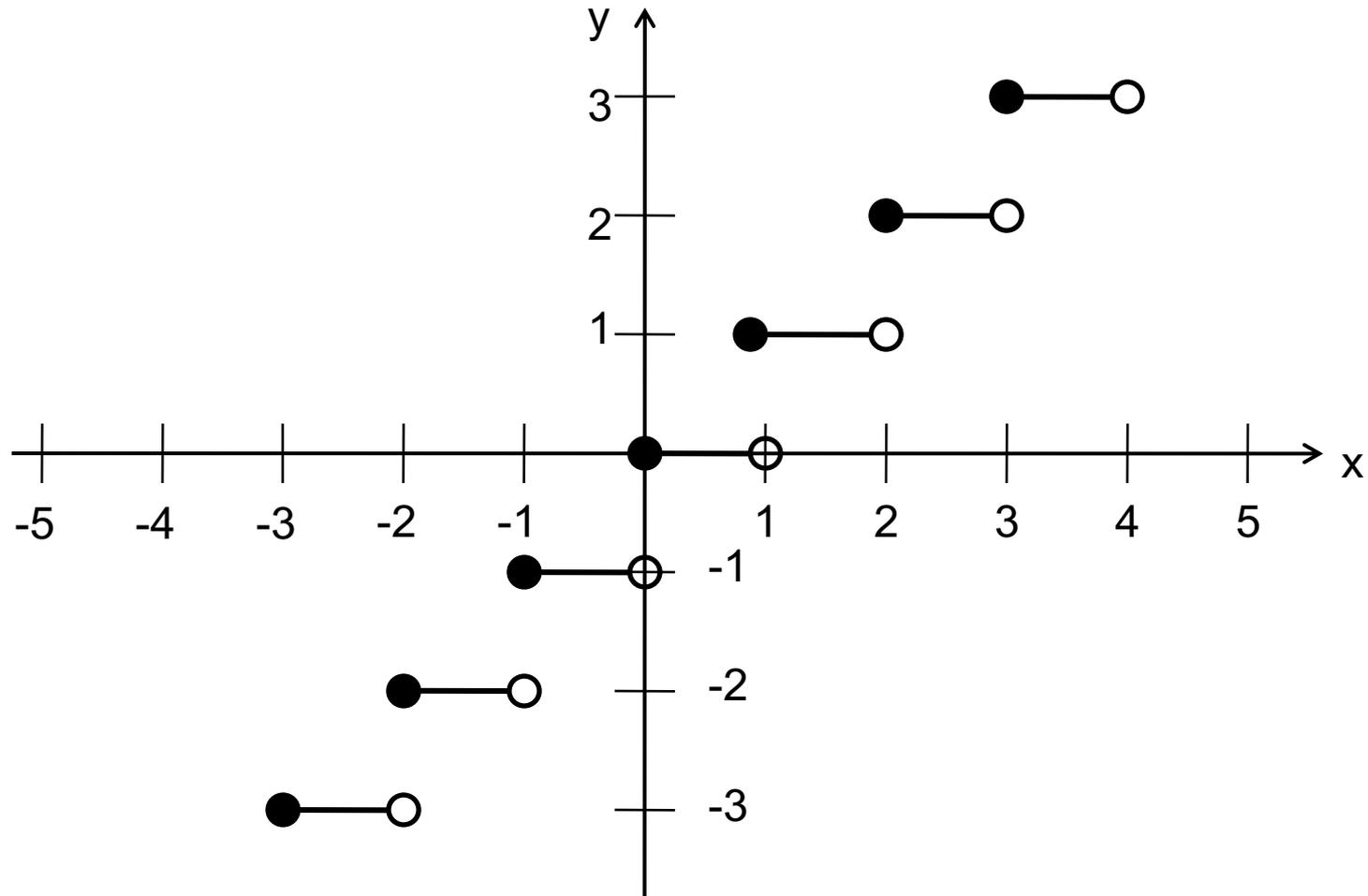$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Important Functions: Floor & Ceiling
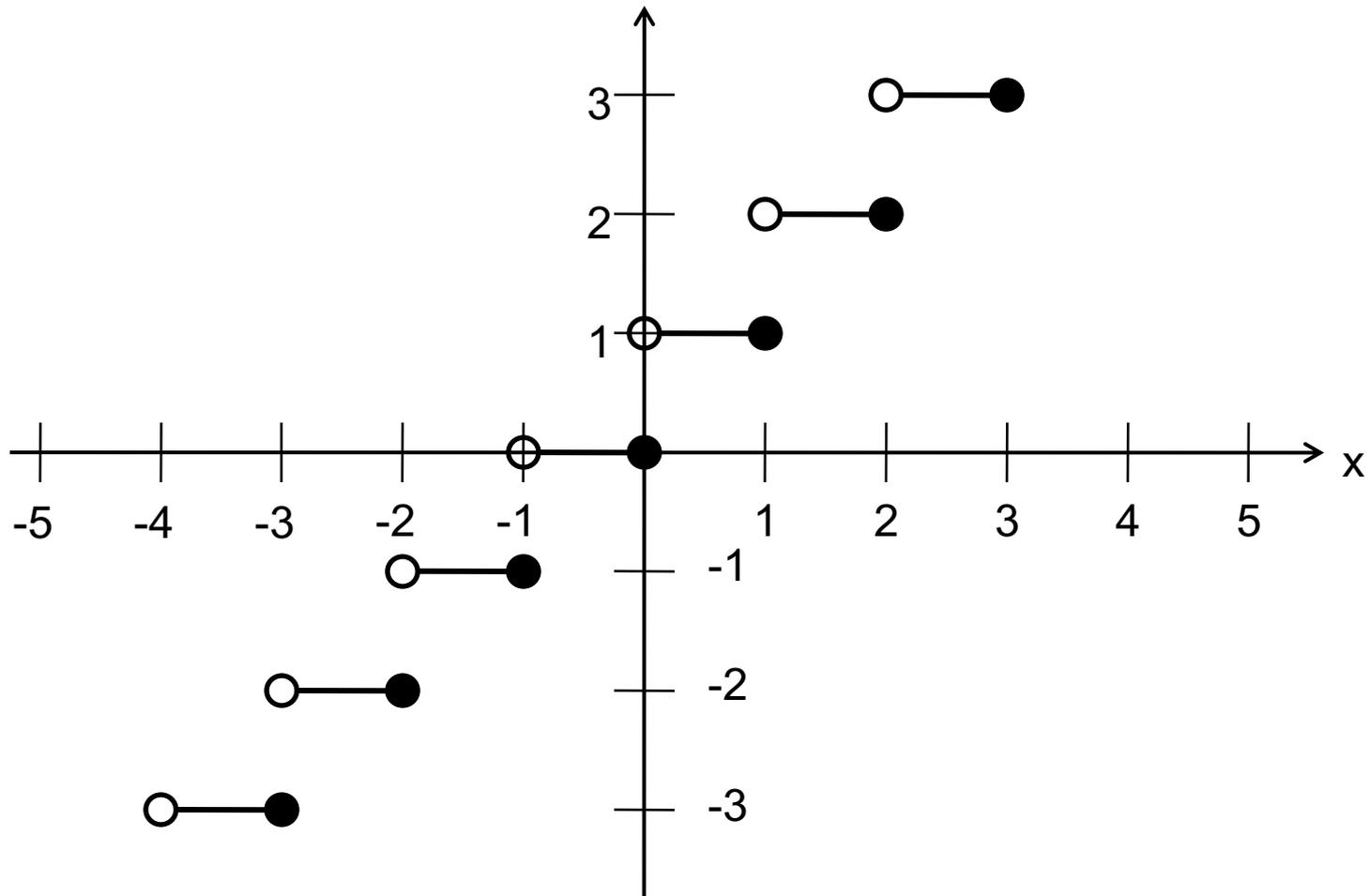
- **Definitions**:
  - The <u>floor function</u>, denoted $\lfloor x \rfloor$, is a function $\mathbb{R} \to \mathbb{Z}$. Its values is the <u>largest integer</u> that is less than or equal to x
  - The ceiling function, denoted $\lceil x \rceil$, is a function $\mathbb{R} \to \mathbb{Z}$. Its values is the <u>smallest integer</u> that is greater than or equal to x
- In LaTex: $\lceil$, $\rceil$, $\rfloor$, $\lfloor$

Important Functions: Floor

Important Functions: Ceiling

Important Function: Factorial

- The factorial function gives us the number of permutations (that is, uniquely ordered arrangements) of a collection of n objects

- **Definition**: The <u>factorial</u> function, denoted n!, is a function $\mathbb{N} \rightarrow \mathbb{N}^+$. Its value is the <u>product</u> of the n positive integers

$$n! = \Pi_{i=1}^{n}\, i = 1{\cdot}2{\cdot}3{\cdot}\ldots{\cdot}(n\text{-}1){\cdot}n$$

Factorial Function & Stirling's Approximation

- The factorial function is defined on a discrete domain

- In many applications, it is useful a continuous version of the function (say if we want to differentiate it)

- To this end, we have the Stirling's formula

$$n! = \sqrt{2\pi n}\, (n/e)^n$$

Summary

- Properties
  - One-to-one (injective), onto (surjective), one-to-one correspondence (bijective)
- Inverse functions (examples)
- Operators
  - Composition, Equality
- Important functions
  - identity, absolute value, floor, ceiling, factorial

Next class

- Topic: Sequences
- Pre-class reading: Chap 2.4