Lecture 10: Sequences and Summations (2)
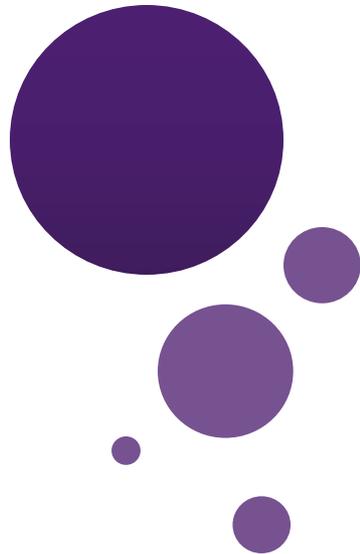
Dr. Chengjiang Long

Computer Vision Researcher at Kitware Inc.

Adjunct Professor at SUNY at Albany.

Email: **clong2@albany.edu**

UNIVERSITY
AT ALBANY

State University of New York

Outline

- Special sequences
- Sum of the elements of a sequence

Outline

- **Special sequences**
- Sum of the elements of a sequence

Sequences

- **Definition**: A <u>sequence</u> is a function from a subset of integers to a set S. We use the notation(s):

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

- Each a_n is called the nth <u>term</u> of the sequence
- We rely on the context to distinguish between a sequence and a set, although they are distinct structures

Geometric Sequence

- **Definition**: A <u>geometric progression</u> is a sequence of the form

$$a, aq, aq^2, aq^3, \ldots, aq^n, \ldots$$

  Where:
  - $a \in R$ is called the <u>initial term</u>
  - $q \in R$ is called the <u>common ratio</u>

- A geometric progression is a <u>discrete</u> analogue of the exponential function

$$f(x) = aq^x$$

Arithmetic Sequence

- **Definition**: An <u>arithmetic progression</u> is a sequence of the form

    a, a+d, a+2d, a+3d, …, a+nd, …

    Where:

  - a$\in R$ is called the <u>initial term</u>
  - d$\in R$ is called the <u>common difference</u>

- An arithmetic progression is a <u>discrete</u> analogue of the linear function

    f(x) = dx+a

Arithmetic Series

Consider an arithmetic series $a_1, a_2, a_3, \ldots, a_n$. If the common difference $(a_{i+1} - a_1) = d$, then we can compute the kth term a_k as follows:

$$a_2 = a_1 + d$$

$$a_3 = a_2 + d = a_1 + 2d$$

$$a_4 = a_3 + d = a_1 + 3d$$

$$a_k = a_1 + (k-1).d$$

Fibonacci Sequence

- Sequences can be neither geometric or arithmetic

  - $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
    - Alternative, $F(n) = F(n-1) + F(n-2)$
  - Each term is the sum of the previous two terms
  - Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
  - This is the Fibonacci sequence

  - Full formula:
  $$F(n) = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5} \cdot 2^n}$$

$$\lim_{n \to \infty} \frac{F(n+1)}{F(n)} = \phi = \frac{\sqrt{5}+1}{2} = 1.618933989$$

Golden ratio.

Harmonic Sequence

- The sequence: $\{h_n\}_{n=1}^{\infty}$ = 1/n

  is known as the **harmonic** sequence

- The sequence is simply:

$$1, 1/2, 1/3, 1/4, 1/5, \ldots$$

- This sequence is particularly interesting because its summation is divergent:

$$\sum_{n=1}^{\infty} (1/n) = \infty$$

Some useful sequences

- $n^2 = 1, 4, 9, 16, 25, 36, \ldots$
- $n^3 = 1, 8, 27, 64, 125, 216, \ldots$
- $n^4 = 1, 16, 81, 256, 625, 1296, \ldots$
- $2^n = 2, 4, 8, 16, 32, 64, \ldots$
- $3^n = 3, 9, 27, 81, 243, 729, \ldots$
- $n! = 1, 2, 6, 24, 120, 720, \ldots$

Outline

- Special sequences
- **Sum of the elements of a sequence**

Summation

- You should be by now familiar with the summation notation:

$$\sum_{i=m}^{n} a_i \;=\; a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

Here

- i is the index of the summation
- m is the lower limit
- n is the upper limit

- Often times, it is useful to change the lower/upper limits, which can be done in a straightforward manner (although we must be very careful):

$$\sum_{i=1}^{n} a_i = \sum_{i=0}^{n-1} a_{i+1}$$

Summation

- A summation:

$$\sum_{j=m}^{n} a_j \quad \text{or} \quad \sum_{j=m}^{n} a_i$$

upper limit

lower limit

index of summation

- is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
        sum += a(j);
```

Summation of Geometric Sequence

With 5 terms of the general geometric sequence, we have

$$S_5 = a + ar + ar^2 + ar^3 + ar^4$$

TRICK Multiply by *r:*

$$rS_5 = ar + ar^2 + ar^3 + ar^4 + ar^5$$

Subtracting the expressions gives

$$S_5 - rS_5 = \quad a + ar + ar^2 + ar^3 + ar^4$$
$$- \quad ar + ar^2 + ar^3 + ar^4 + ar^5$$

Move the lower row 1 place to the right and subtract

Summation of Geometric Sequence

With 5 terms of the general geometric sequence, we have

$$S_5 = a + ar + ar^2 + ar^3 + ar^4$$

TRICK Multiply by *r:*

$$rS_5 = ar + ar^2 + ar^3 + ar^4 + ar^5$$

Subtracting the expressions gives

$$S_5 - rS_5 = a + ar + ar^2 + ar^3 + ar^4$$
$$- ar + ar^2 + ar^3 + ar^4 + ar^5$$

$$S_5 - rS_5 = a \qquad\qquad - ar^5$$

Summation of Geometric Sequence

So,
$$S_5 - rS_5 = a - ar^5$$

Take out the common factors
$$S_5(1-r) = a(1 - r^5)$$

and divide by $(1 - r)$
$$\Rightarrow \quad S_5 = \frac{a(1 - r^5)}{1-r}$$

Similarly, for n terms we get
$$S_n = \frac{a(1 - r^n)}{1-r}$$

Summation of Geometric Sequence

The formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

gives a negative denominator if $r > 1$

Instead, we can use

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Example

Find the sum of the first 20 terms of the geometric series, leaving your answer in index form $\mathbf{2 - 6 + 18 - 54 + \ .\ \ .\ \ .}$

Solution: $\quad a = 2, \quad r = \dfrac{\overset{-3}{\cancel{-6}}}{\underset{1}{\cancel{2}}} = -3$

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \Rightarrow \quad S_{20} = \frac{2\left(1 - (-3)^{20}\right)}{1 - (-3)}$$

We'll simplify this answer without using a calculator

Example

$$\Rightarrow \quad S_{20} = \frac{2\left(1-(-3)^{20}\right)}{1-(-3)}$$

There are 20 minus signs here and 1 more outside the bracket!

$$= \frac{{}^{1}2\left(1-3^{20}\right)}{{}^{2}4}$$

$$= \frac{1-3^{20}}{2}$$

Sum of arithmetic series

What is the sum of this series?

$$\sum_{p=1}^{50}(73-2p)$$

$$= 71 + 69 + 67 + \ldots + (-25) + (-27)$$

- Write the first three terms and the last two terms of the following arithmetic series.

Sum of arithmetic series

What is the sum of these terms?

$$71 + 69 + 67 + \ldots + (-25) + (-27)$$ Written 1st to last.

$$(-27) + (-25) + \ldots + 67 + 69 + 71$$ Written last to 1st.

$$44 + 44 + 44 + \ldots + 44 + 44 + 44$$ **Add Down**

50 Terms

$$= \frac{50(71 + (-27))}{2}$$

71 + (-27) Each sum is the same.

$$= 1100$$

Sum of arithmetic series

In general

$$a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d)$$

$$(a_1 + (n-1)d) + \ldots + (a_1 + 2d) + (a_1 + d) + a_1$$

$$\overline{}$$

$$\left(a_1 + \left[a_1 + (n-1)d\right]\right) + \left(a_1 + \left[a_1 + (n-1)d\right]\right) + \ldots + \left(a_1 + \left[a_1 + (n-1)d\right]\right)$$

$$S = \frac{n(a_1 + a_n)}{2} \qquad \begin{cases} S = \text{Sum} \\ n = \text{Number of Terms} \\ a_1 = \text{First Term} \\ a_n = \text{Last Term} \end{cases}$$

Example

$$151 + 147 + 143 + 139 + \ldots + (-5)$$

What term is -5?

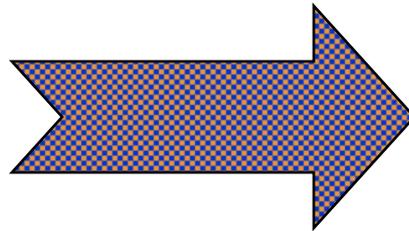$$S = \frac{n(a_1 + a_n)}{2}$$

$$a_n = a_1 + (n-1)d$$

$$-5 = 151 + (n-1)(-4)$$

$$n = 40$$

$$n = 40$$
$$a_1 = 151$$
$$a_{40} = -5$$

$$S = \frac{40(151 + -5)}{2}$$

$$S = 2920$$

Sum of arithmetic series

Alternate formula for the sum of an Arithmetic Series.

Substitute $a_n = a_1 + (n-1)d$

$$S = \frac{n(a_1 + a_n)}{2}$$

$$S = \frac{n(a_1 + a_1 + (n-1)d)}{2}$$

$$S = \frac{n(2a_1 + (n-1)d)}{2}$$

$$\begin{cases} n = \#\text{ of Terms} \\ a_1 = 1\text{st Term} \\ d = \text{Difference} \end{cases}$$

Solve this

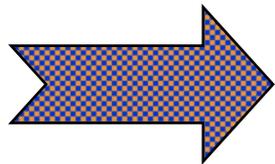$$\sum_{j=0}^{36} \left(2.25 + 0.75j\right) = 2.25 + 3 + 3.73 + 4.5 + \ldots$$

$$S = \frac{n\left(2a_1 + (n-1)d\right)}{2}$$

It is not convenient to find the last term.

$n = 37$

$a_1 = 2.25$

$d = 0.75$

$$S = \frac{37\left(2(2.25) + (37-1)(0.75)\right)}{2}$$

$$S = 582.75$$

Evaluating sequences

$$\sum_{k=1}^{5} (k+1)$$

- $2 + 3 + 4 + 5 + 6 = 20$

$$\sum_{j=0}^{4} (-2)^j$$

- $(-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$

$$\sum_{i=1}^{10} 3$$

- $3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30$

$$\sum_{j=0}^{8} \left(2^{j+1} - 2^j\right)$$

- $(2^1-2^0) + (2^2-2^1) + (2^3-2^2) + \ldots (2^{10}-2^9) = 511$
  - Note that each term (except the first and last) is cancelled by another term

Series

- When we take the <u>sum of a sequence</u>, we get a <u>series</u>

- We have already seen a closed form for geometric series

- Some other useful closed forms include the following:

  - $\sum_{i=k}^{u} 1$ = u-k+1, for k≤u

  - $\sum_{i=k}^{u} i$ = n(n+1)/2

  - $\sum_{i=1}^{n} i^2$ = n(n+1)(2n+1)/6

  - $\sum_{i=1}^{n} i^k \approx$ n**k+1**/(k+1)

Infinite Series

- Although we will mostly deal with finite series (i.e., an upper limit of n for fixed integer), inifinite series are also useful

- Consider the following geometric series:
  - $\Sigma_{n=0}(1/2^n) = 1 + 1/2 + 1/4 + 1/8 + \ldots$ converges to 2
  - $\Sigma_{n=0}(2^n) = 1 + 2 + 4 + 8 + \quad \ldots$ does not converge

- However note: $\Sigma_{n=0}(2^n) = 2^{n+1} - 1$ (a=1, q=2)

Evaluating sequences

- Let S = { 1, 3, 5, 7 }

- What is $\sum_{j \in S} j$
  - $1 + 3 + 5 + 7 = 16$
- What is $\sum_{j \in S} j^2$
  - $1^2 + 3^2 + 5^2 + 7^2 = 84$
- What is $\sum_{j \in S} (1/j)$
  - $1/1 + 1/3 + 1/5 + 1/7 = 176/105$
- What is $\sum_{j \in S} 1$
  - $1 + 1 + 1 + 1 = 4$

Can you evaluate this?

$$\sum_{i=1}^{n} \frac{1}{k(k+1)}$$

Here is the trick. Note that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Does it help?

Double Summation

- Like a nested for loop

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij$$

- Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
        for ( int j = 1; j <= 3; j++ )
                sum += i*j;
```

Solve the following

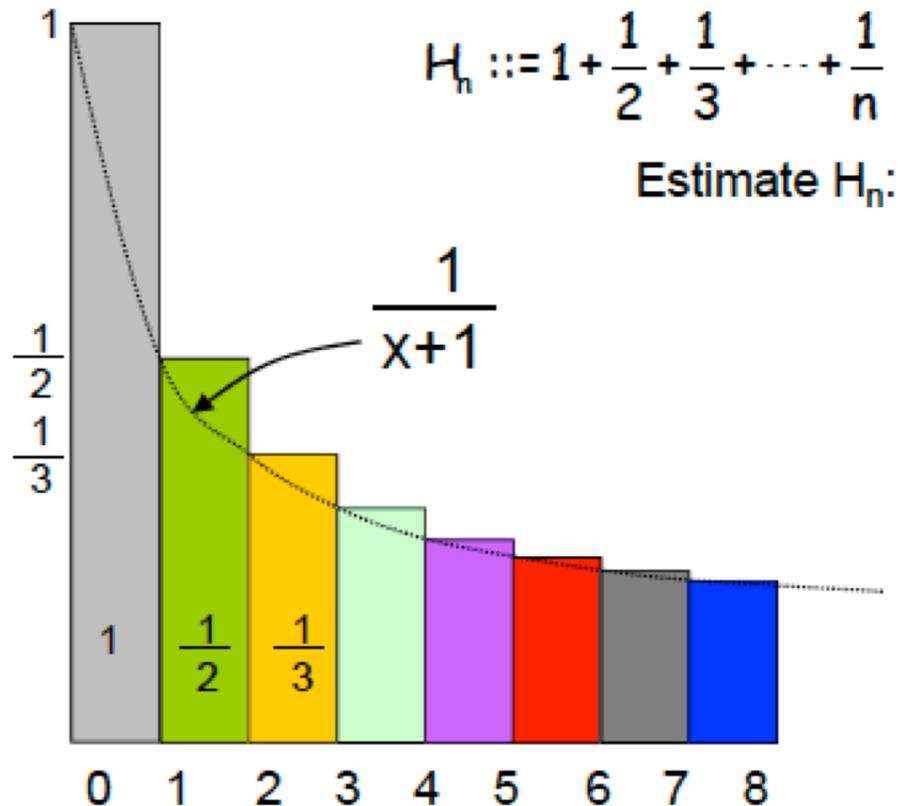$$1 + 1/2 + 1/4 + 1/8 + \cdots = \sum_{i=0}^{\infty} (1/2)^i$$

$$0.999999999\ldots = 0.9 \sum_{i=0}^{\infty} (1/10)^i$$

$$1 - 1/2 + 1/4 - 1/8 + \cdots = \sum_{i=0}^{\infty} (-1/2)^i$$

$$1 + 2 + 4 + 8 + \cdots + 2^{n-1} = \sum_{i=0}^{n-1} 2^i$$

$$1 + 3 + 9 + 27 + \cdots + 3^{n-1} = \sum_{i=0}^{n-1} 3^i$$

Sum of harmonic series

$$H_n ::= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Estimate H_n:

$$\frac{1}{x+1}$$

$$\int_0^n \frac{1}{x+1}\,dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

$$\int_1^{n+1} \frac{1}{x}\,dx \leq H_n$$

$$\ln(n+1) \leq H_n$$

Products

$$\prod_{i=1}^{n} a_i := a_1 \cdot a_2 \cdots a_n$$

$$\prod_{k=1}^{5} k^2$$

$$\prod_{k=1}^{n} \frac{k}{k+1}$$

$$\prod_{k=1}^{n} 2^k$$

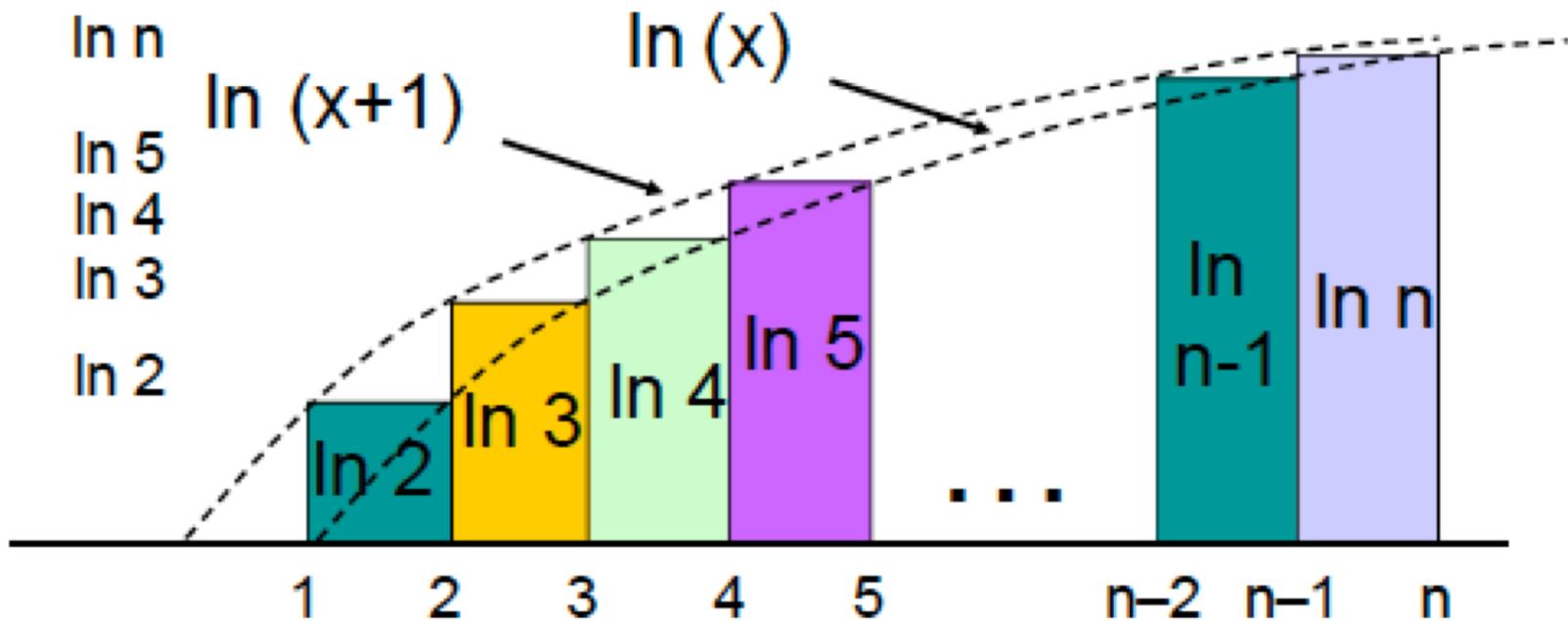# Dealing with Products

Factorial defines a **product**:

$$n! = 1 \cdot 2 \cdot 3 \cdot n = \prod_{i=1}^{n} i$$

> How to estimate n!?

Turn product into a **sum** taking logs:

$$\ln(n!) = \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n)$$

$$= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n)$$

$$= \sum_{i=1}^{n} \ln(i)$$

Factorial

$$\int_1^n \ln(x)\,dx \leq \sum_{i=1}^n \ln(i) \leq \int_0^n \ln(x+1)\,dx$$

Factorial

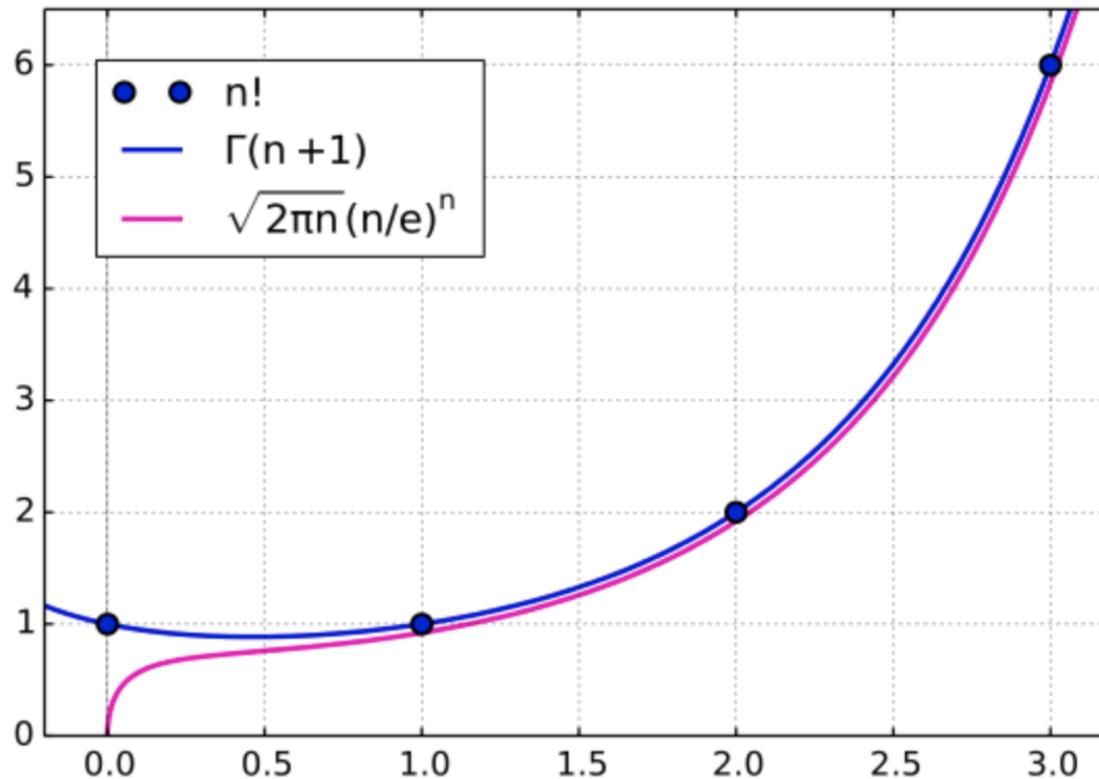$$\int_1^n \ln(x)\, dx \le \sum_{i=1}^{n} \ln(i) \le \int_0^n \ln(x+1)\, dx$$

Reminder: $\int \ln x\, dx = x \ln\left(\dfrac{x}{e}\right)$

$$n \ln(n/e) + 1 \le \sum \ln(i) \le (n+1) \ln((n+1)/e) + 1$$

so guess: $\displaystyle\sum_{i=1}^{n} \ln(i) \approx \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{e}\right)$

Stirling's formula

$$\sum_{i=1}^{n} \ln(i) \approx \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{e}\right)$$

➡️ exponentiating: $n! \approx \sqrt{n/e}\left(\frac{n}{e}\right)^n$

Next class

- Topic: Cardinality of Sets
- Pre-class reading: Chap 2.5