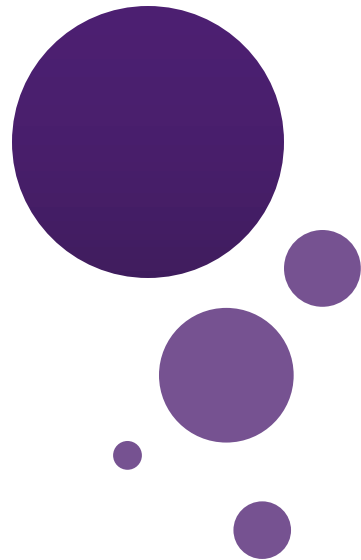




UNIVERSITY
AT ALBANY

State University of New York

Lecture 25: The Pigeonhole Principle, Permutations and Combinations



Dr. Chengjiang Long
Computer Vision Researcher at Kitware Inc.
Adjunct Professor at SUNY at Albany.
Email: clong2@albany.edu

Outline

- The Pigeonhole Principle
- Permutations and Combinations

Outline

- **The Pigeonhole Principle**
- Permutations and Combinations

Pigeonhole Principle

Motivation:

The mapping of n objects to m buckets

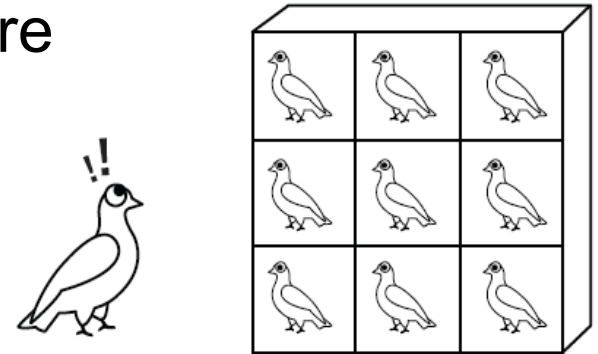
E.g. Hashing.

The principle is used for proofs of certain complexity derivation.

Pigeonhole Principle

Pigeonhole Principle: If n pigeonholes are occupied by $n + 1$ or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.

THE PIGEONHOLE PRINCIPLE



Remark: The principle is obvious. No simpler fact or rule to support or prove it.

Generalized Pigeonhole Principle: If n pigeonholes are occupied by $kn + 1$ pigeons, then at least one pigeonhole is occupied by $k + 1$ or more pigeons.

Example 1: Birthmonth

- In a group of 13 people, we have 2 or more who are born in the same month.

# pigeons	# holes	At least # born in the same month
13	12	2 or more
20	12	2 or more
121	12	11 or more
65	12	6 or more
111	12	10 or more
$\geq k+1$	n	$k+1$ or more

Example 2: Handshaking

Given a group of n people ($n > 1$), each shakes hands with some (a nonzero number of) people in the group. We can find at least two who shake hands with the same number of people.

Proof:

Number of pigeons (number of people): n

Number of pigeonholes (range of number of handshakes):
 $n-1$

Example 3: Cast in theater

A theater performs 7 plays in one season. There are 15 women. Then some play has at least 3 women in its cast.

Number of pigeons: 15

Number of pigeonholes: 7

$$k \cdot n + 1 = 2 \cdot 7 + 1$$

3 or more pigeons in the same pigeonhole

Example 4: Pairwise difference

Given 8 different natural numbers, none greater than 14. Show that at least three pairs of them have the same difference.

Try a set: 1, 2, 3, 7, 9, 11, 12, 14

Difference of 12 and 14 = 2.

Same for 9 and 11; 7 and 9; 1 and 3.

In this set, there are four pairs that all have the same difference.

Example 4: Pairwise difference

Given 8 different natural numbers, none greater than 14. Show that at least three pairs of them have the same difference.

Try a set: 1, 2, 3, 7, 9, 11, 12, 14

Difference of 12 and 14 = 2.

Same for 9 and 11; 7 and 9; 1 and 3.

In this set, there are four pairs that all have the same difference.

Proof:

pigeons (different pairs: $C(8,2) = 8*7/2$): 28

pigeonholes (14-1): 13

Since $28 \geq k*n + 1 = 2*13 + 1$, we have

3 or more pigeons in the same pigeonhole.

Remark

- Pigeonhole principle has applications to assignment and counting.
- The usage of the principle relies on the identification of the pigeons and the pigeonholes.

Outline

- The Pigeonhole Principle
- **Permutations and Combinations**

Permutations

Definition: A **permutation** of a set S is a sequence that contains every element of S exactly once.

For example, here are all six permutations of the set $\{a, b, c\}$:

(a, b, c) (a, c, b) (b, a, c)
 (b, c, a) (c, a, b) (c, b, a)

Ordering is important here.

How many permutations of an n -element set are there?

You can think of a permutation as a ranking of the elements.

So the above question is asking how many rankings of an n -element set.

Permutations

How many permutations of an n-element set are there?

- There are n choices for the first element.
- For each of these, there are n – 1 remaining choices for the second element.
- For every combination of the first two elements, there are n – 2 ways to choose the third element, and so forth.
- Thus, there are a total of
$$n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 3 \cdot 2 \cdot 1 = n!$$
 permutations of an n-element set. This is called n **factorial**.

Stirling's formula (optional):

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Example

Suppose each digit is an element in $\{1,2,3,4,5,6,7,8,9\}$.

How many 9-digit numbers are there where each nonzero digit appears once?

Each such number corresponds to a permutation of 123456789,
and each permutation corresponds to such a number.

So the numbers of such numbers is equal to
the number of permutations of $\{1,2,3,4,5,6,7,8,9\}$.

Hence there are exactly $9!$ such numbers.

Alternatively, one can use the generalized product rule
directly to obtain the same result.

Combinations

How many subsets of size k of an n -element set?

Consider the set $\{1,2,3,4,5\}$ where $n=5$.

If $k=2$, then there are 10 possible subsets of size 2,
i.e. $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}$.

If $k=3$, then there are also 10 possible subsets of size 3,
i.e. $\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}$
 $\{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}$

Ordering is NOT important here.

Combinations

How many subsets of size k of an n -element set?

- There are n choices for the first element.
- For each of these, there are $n - 1$ remaining choices for the second element.
- There are $n - k + 1$ remaining choices for the last element.
- Thus, there are a total of
$$n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot (n - k + 1)$$
 to choose k elements.

So far we counted the number of ways to choose k elements,
when the ordering is important.

e.g. $\{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, $\{3,2,1\}$ will be counted as 6 different ways.

Combinations

How many subsets of size k of an n -element set?

We form the subsets by picking one element at a time.

- There are n choices for the first element.
- For each of these, there are $n - 1$ remaining choices for the second element.
- There are $n - k + 1$ remaining choices for the last element.
- Thus, there are a total of
$$n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot (n - k + 1)$$
 to choose k elements.

So far we counted the number of ways to choose k elements,
when the ordering is important.

Combinations

How many subsets of size k of an n -element set?

- Thus, there are a total of $n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot (n - k + 1)$ ways to choose k elements, when the ordering is important.

How many different ordering of k elements are (over)-counted?

e.g. If we are forming subsets of size 3, then

$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$

are counted as 6 different ways if the ordering is important.

In general, each subset of size k has $k!$ different orderings,
and so each subset is counted $k!$ times in the above way of choosing k elements.

Combinations

How many subsets of size k of an n -element set?

- Thus, there are a total of $n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot (n - k + 1)$ ways to choose k elements, when the ordering is important.
- Each subset is counted, but is counted $k!$ times, because each subset contributes $k!$ different orderings to the above.
- So, when the ordering is not important, the answer is:

This is the
shorthand for
“ n choose k ”

$$\binom{n}{k} = \frac{n \cdot (n - 1) \cdot \dots \cdot (n - k + 1)}{k!} = \frac{n!}{(n - k)!k!}$$

Example: Team Formation

There are m boys and n girls.

How many ways are there to form a team with 3 boys and 3 girls?

There are $\binom{m}{3}$ choices of 3 boys and $\binom{n}{3}$ choices for 3 girls.

So by the product rule there are $\binom{m}{3} \binom{n}{3}$ choices of such a team.

If $m < 3$ or $n < 3$, then the answer should be zero.
Don't worry. We don't like to trick you this way.

Example: Bit Strings with k Zeros

How many n-bit sequences contain k zeros and (n – k) ones?

We can think of this problem as choosing k positions (out of the n possible positions) and set them to zeroes and set the remaining positions to ones.

So the above question is asking the number of possible positions of the k zeros, and the answer is:

$$\binom{n}{k}$$

Example: Unbalanced Bit Strings

We say a bit string is unbalanced if there are more ones than zeroes or more zeros than ones.

How many n -bit strings are unbalanced?

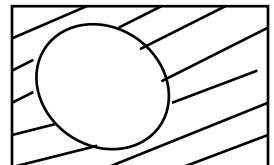
If n is odd, then every n -bit string is unbalanced, and the answer is 2^n .

If n is even, then the number of balanced strings is

$\binom{n}{n/2}$ by choosing $n/2$ positions to zeroes.

So the number of unbalanced n -bit strings is equal to the number of all n -bit strings minus the number of balanced strings,

and so the answer is $2^n - \binom{n}{n/2}$ (counting the complement)



Poker Hands

There are 52 cards in a deck.
Each card has a suit and a value.

4 suits (   )

13 values (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)

Five-Card Draw is a card game in which each player
is initially dealt a hand, a subset of 5 cards.

How many different hands?

$$\binom{52}{5} = 2598960$$

Example 1: Four of a Kind

A Four-of-a-Kind is a set of four cards with the same value.

$$\begin{array}{l} \{ 8\spadesuit, 8\diamondsuit, Q\heartsuit, 8\heartsuit, 8\clubsuit \} \\ \{ A\clubsuit, 2\clubsuit, 2\heartsuit, 2\diamondsuit, 2\spadesuit \} \end{array}$$

How many different hands contain a Four-of-a-Kind?

One way to do this is to first map the problem into a problem of counting sequences.

Example 1: Four of a Kind

A hand with a Four-of-a-Kind is completely described by a sequence specifying:

1. The value of the four cards.
2. The value of the extra card.
3. The suit of the extra card.

$$\begin{array}{lcl} (8, Q, \heartsuit) & \leftrightarrow & \{ 8\spadesuit, 8\diamondsuit, 8\heartsuit, 8\clubsuit, Q\heartsuit \} \\ (2, A, \clubsuit) & \leftrightarrow & \{ 2\clubsuit, 2\heartsuit, 2\diamondsuit, 2\spadesuit, A\clubsuit \} \end{array}$$

There are 13 choices for (1), 12 choices for (2), and 4 choices for (3).
By generalized product rule, there are $13 \times 12 \times 4 = 624$ hands.

Only 1 hand in about 4165 has a Four-of-a-Kind!

Example 2: Full House

A **Full House** is a hand with three cards of one value and two cards of another value.

$$\begin{array}{l} \{ 2\spadesuit, 2\clubsuit, 2\diamondsuit, J\clubsuit, J\diamondsuit \} \\ \{ 5\diamondsuit, 5\clubsuit, 5\heartsuit, 7\heartsuit, 7\clubsuit \} \end{array}$$

How many different hands contain a Full House?

Example 2: Full House

There is a bijection between Full Houses and sequences specifying:

1. The value of the triple, which can be chosen in 13 ways.
2. The suits of the triple, which can be selected in $(4\ 3)$ ways.
3. The value of the pair, which can be chosen in 12 ways.
4. The suits of the pair, which can be selected in $(4\ 2)$ ways.

$$\begin{aligned} (2, \{\spadesuit, \clubsuit, \diamondsuit\}, J, \{\clubsuit, \diamondsuit\}) &\leftrightarrow \{ 2\spadesuit, 2\clubsuit, 2\diamondsuit, J\clubsuit, J\diamondsuit \} \\ (5, \{\diamondsuit, \clubsuit, \heartsuit\}, 7, \{\heartsuit, \clubsuit\}) &\leftrightarrow \{ 5\diamondsuit, 5\clubsuit, 5\heartsuit, 7\heartsuit, 7\clubsuit \} \end{aligned}$$

By generalized product rule, there are

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3744$$

Only 1 hand in about 634 has a Full House!

Example 3: Two Pairs

How many hands have **Two Pairs**; that is,
two cards of one value, two cards of another value,
and one card of a third value?

$$\begin{array}{l} \{ 3\diamondsuit, 3\spadesuit, Q\diamondsuit, Q\heartsuit, A\clubsuit \} \\ \{ 9\heartsuit, 9\diamondsuit, 5\heartsuit, 5\clubsuit, K\spadesuit \} \end{array}$$

Example 3: Two Pairs

1. The value of the first pair, which can be chosen in 13 ways.
2. The suits of the first pair, which can be selected $\binom{4}{2}$ ways.
3. The value of the second pair, which can be chosen in 12 ways.
4. The suits of the second pair, which can be selected in $\binom{4}{2}$ ways
5. The value of the extra card, which can be chosen in 11 ways.
6. The suit of the extra card, which can be selected in 4 ways.

$$\text{Number of Two pairs} = 13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4$$

Double
Count!

$$\begin{array}{l} (3, \{\diamond, \spadesuit\}, Q, \{\diamond, \heartsuit\}, A, \clubsuit) \searrow \\ (Q, \{\diamond, \heartsuit\}, 3, \{\diamond, \spadesuit\}, A, \clubsuit) \nearrow \end{array} \quad \{ 3\diamond, 3\spadesuit, Q\diamond, Q\heartsuit, A\clubsuit \}$$

$$\text{So the answer is } \frac{1}{2} \cdot 13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4 = 123552$$

Example 4: Every Suit

How many hands contain at least one card from every suit?

$$\{ 7\Diamond, K\clubsuit, 3\Diamond, A\heartsuit, 2\spadesuit \}$$

1. The value of each suit, which can be selected in $13 \times 13 \times 13 \times 13$ ways.
2. The suit of the extra card, which can be selected in 4 ways.
3. The value of the extra card, which can be selected in 12 ways.

$$(7, K, A, 2, \Diamond, 3) \leftrightarrow \{ 7\Diamond, K\clubsuit, A\heartsuit, 2\spadesuit, 3\Diamond \}$$

$$(7, K, A, 2, \Diamond, 3) \searrow$$

$$\{ 7\Diamond, K\clubsuit, A\heartsuit, 2\spadesuit, 3\Diamond \}$$

Double count!

$$(3, K, A, 2, \Diamond, 7) \nearrow$$

So the answer is $13^4 \times 4 \times 12 / 2 = 685464$

Next class

- Topic: Binomial Coefficients and Identities
- Pre-class reading: Chap 6.4

