Lecture 26: Binomial Coefficients and Identities

Dr. Chengjiang Long
Computer Vision Researcher at Kitware Inc.
Adjunct Professor at SUNY at Albany.
Email: **clong2@albany.edu**

Announcement

- Midterm Exam 2 will be taken on Nov 14th, 2018.
  - One-hour exam like Midterm Exam 1.
  - It covers Chap 3.1 – Chap 6.4, Lecture 13 – Lecture 26.
  - No sheet provided.

Outline

- r-Permutations and r-Combinations
- Binomial coefficients, combinatorial proof
- Inclusion-exclusion principle

Outline

- **r-Permutations and r-Combinations**
- Binomial coefficients, combinatorial proof
- Inclusion-exclusion principle

r-permutations

If n is a positive integer and r is an integer with $1 \le r \le n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of a set with n distinct elements.

- How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

  Solution: $P(100, 3) = 100 \cdot 99 \cdot 98 = 970{,}200.$

- How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

  Solution: *P*(6,6) = 6! = 720
  ABC, D, E, F, G, H

r-combination

- An ***r*-combination** of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

$$P(n, r) = C(n, r) \cdot P(r, r).$$

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!\,(n-r)!}.$$

$$C(n, r) = \frac{n!}{r!\,(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

Example

- How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

$$C(52,5) = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2{,}598{,}960.$$

$$C(52,47) = \frac{52!}{47!5!}$$

Let n and r be nonnegative integers with $r \leq n$. Then $C(n,r) = C(n, n-r)$.

Outline

- r-Permutations and r-Combinations
- **Binomial coefficients, combinatorial proof**
- Inclusion-exclusion principle

Binomial Theorem

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

We can compute the coefficients by simple counting arguments.

$$(1+x)^n = \underbrace{(1+x)(1+x)(1+x)\cdots\cdots(1+x)}_{\text{n times}}$$

Each term corresponds to selecting 1 or x from each of the n factors.

c_k is number of terms with exactly k x's are selected from n factors.

$$c_k = \binom{n}{k}$$

Binomial Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Binomial Coefficients

In general we have the following identity:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

When x=1, y=1, it says that $2^n = \sum_{i=0}^{n} \binom{n}{i}$

When x=1, y=-1, it says that

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} + \dots$$

$$\Rightarrow \sum_{i \text{ odd}} \binom{n}{i} = \sum_{j \text{ even}} \binom{n}{j}$$

Proving Identities

$$\binom{n}{k} = \binom{n}{n-k}$$

Direct proof: $$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Combinatorial proof: Number of ways to choose k items from n items

= number of ways to choose n-k items from n items

Finding a Combinatorial Proof

A **combinatorial proof** is an argument that establishes an algebraic fact by relying on counting principles.

Many such proofs follow the same basic outline:

1. Define a set S.
2. Show that |S| = n by counting one way.
3. Show that |S| = m by counting another way.
4. Conclude that n = m.

Double counting

Proving Identities

Pascal's Formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Direct proof:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!}$$

$$= \frac{n!(n+1)}{k!(n-k+1)!}$$

$$= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$$

Proving Identities

Pascal's Formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Combinatorial proof:

- The LHS is number of ways to choose k elements from n+1 elements.
- Let the first element be x.
- If we choose x, then we need to choose k-1 elements from the remaining n elements, and number of ways to do so is $\binom{n}{k-1}$
- If we don't choose x, then we need to choose k elements from the remaining n elements, and number of ways to do so is $\binom{n}{k}$
- This partitions the ways to choose k elements from n+1 elements, therefore $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$
$$\ldots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$
$$\ldots$$

(b)

Combinatorial Proof

$$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Consider we have 2n balls, n of them are red, and n of them are blue.

The RHS is number of ways to choose n balls from the 2n balls.
To choose n balls, we can

- choose 0 red ball and n blue balls, number of ways = $\binom{n}{0}\binom{n}{n}$
- choose 1 red ball and n-1 blue balls, number of ways = $\binom{n}{1}\binom{n}{n-1}$
- …
- choose i red balls and n-i blue balls, number of ways = $\binom{n}{i}\binom{n}{n-i}$
- …
- choose n red balls and 0 blue ball, number of ways = $\binom{n}{n}\binom{n}{0}$

Hence number of ways to choose n balls is also equal to $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$

Another Way to Combinatorial Proof (Optional)

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$$

We can also prove the identity by comparing a coefficient of two polynomials.

Consider the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$

Consider the coefficient of x^n in these two polynomials.

Clearly the coefficient of x^n in $(1+x)^{2n}$ is equal to the RHS.

$$(1+x)^n(1+x)^n = (\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{n}x^n)(\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{n}x^n)$$

So the coefficient of x^n in $(1+x)^n(1+x)^n$ is equal to the LHS.

More Combinatorial Proof

$$\sum_{r=0}^{n} \binom{n}{r}\binom{2n}{n-r} = \binom{3n}{n}$$

Let S be all n-card hands that can be dealt from a deck containing
n red cards (numbered 1, . . . , n) and 2n black cards (numbered 1, . . . , 2n).

The right hand side = # of ways to choose n cards from these 3n cards.

The left hand side = # of ways to choose r cards from red cards x
# of ways to choose n-r cards from black cards
= # of ways to choose n cards from these 3n cards
= the right hand side.

Exercises

Prove that

$$3^n = 1 + 2n + 4\binom{n}{2} + 8\binom{n}{3} + \ldots + 2^k\binom{n}{k} + \ldots + 2^n\binom{n}{n}$$

Give a combinatorial proof of the following identify.

$$\binom{n}{0}\binom{2n}{n} + \binom{n}{1}\binom{2n}{n-1} + \ldots + \binom{n}{k}\binom{2n}{n-k} + \ldots + \binom{n}{n}\binom{2n}{0} = \binom{3n}{n}$$

Can you give a direct proof of it?

Quick Summary

We have studied how to determine the size of a set directly.

The basic rules are the sum rule, product rule, and the generalized product rule.

We apply these rules in counting permutations and combinations,

which are then used to count other objects like poker hands.

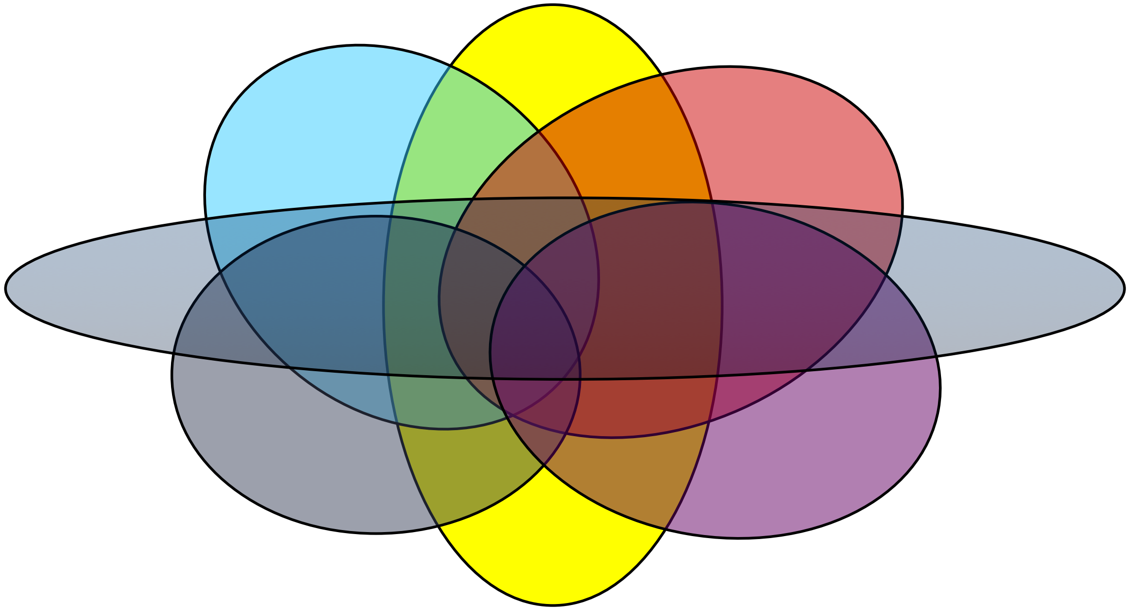
Then we prove the binomial theorem and study combinatorial proofs of identities.

Finally we learn the inclusion-exclusion principle and see some applications.

Later we will learn how to count “indirectly” by “mapping”.

Outline

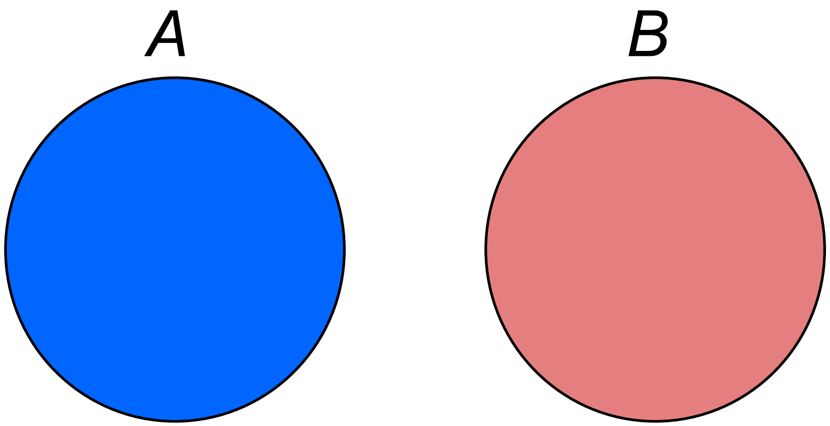
- r-Permutations and r-Combinations
- Binomial coefficients, combinatorial proof
- **Inclusion-exclusion principle**



Sum Rule

If sets A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

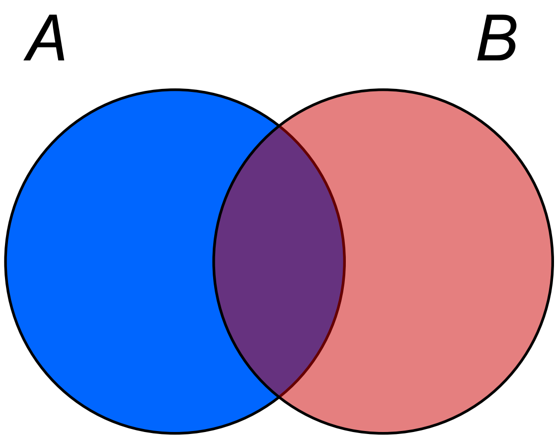


What if A and B are not disjoint?

Inclusion-Exclusion (2 Sets)

For two arbitrary sets *A* and *B*

$$|A \cup B| = |A| + |B| - |A \cap B|$$



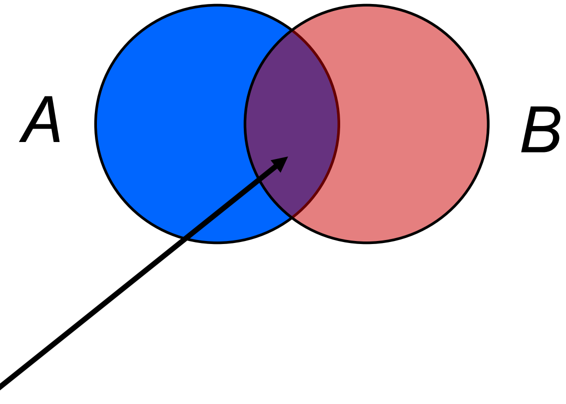
Inclusion-Exclusion (2 Sets)

Let S be the set of integers from 1 through 1000 that are multiples of 3 or multiples of 5.

Let A be the set of integers from 1 to 1000 that are multiples of 3.

Let B be the set of integers from 1 to 1000 that are multiples of 5.

It is clear that S is the union of A and B, but notice that A and B are not disjoint.



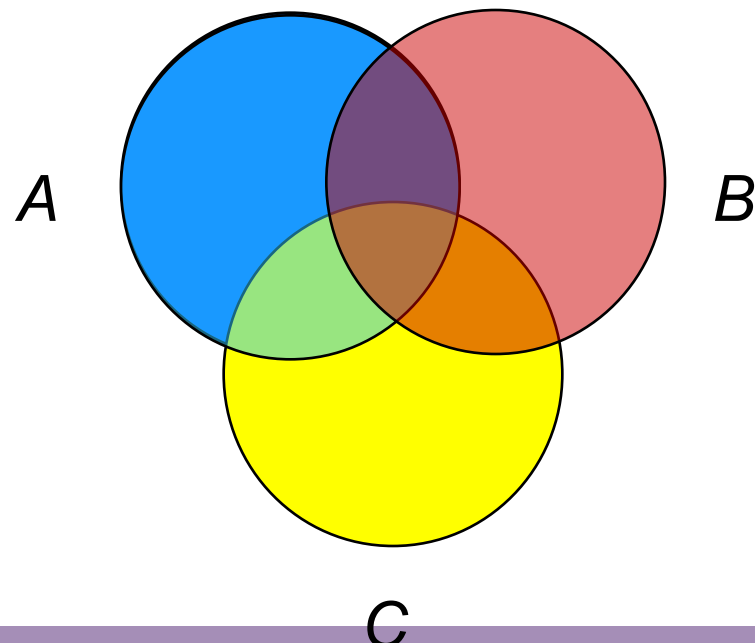
$|A| = 1000/3 = 333$ $|B| = 1000/5 = 200$

$A \cap B$ is the set of integers that are multiples of 15, and so $|A \cap B| = 1000/15 = 66$

So, by the inclusion-exclusion principle, we have $|S| = |A| + |B| - |A \cap B| = 467$.

Inclusion-Exclusion (3 Sets)

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Inclusion-Exclusion (3 Sets)

From a total of 50 students:

$|A| \rightarrow$ 30 know Java

$|B| \rightarrow$ 18 know C++

$|C| \rightarrow$ 26 know C#

$|A \cap B| \rightarrow$ 9 know both Java and C++

$|A \cap C| \rightarrow$ 16 know both Java and C#

$|B \cap C| \rightarrow$ 8 know both C++ and C#

$|A \cup B \cup C| \rightarrow$ 47 know at least one language.

How many know none?

How many know all? $\uparrow$ $|A \cap B \cap C|$

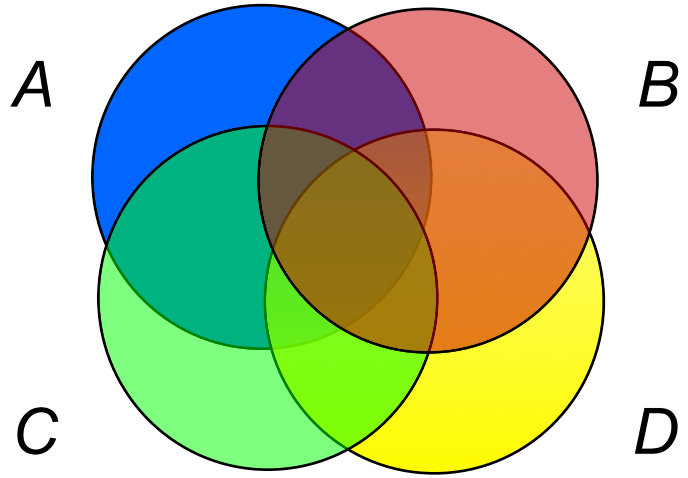
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$47 = 30 + 18 + 26 - 9 - 16 - 8 + |A \cap B \cap C|$$

$$|A \cap B \cap C| = 6$$

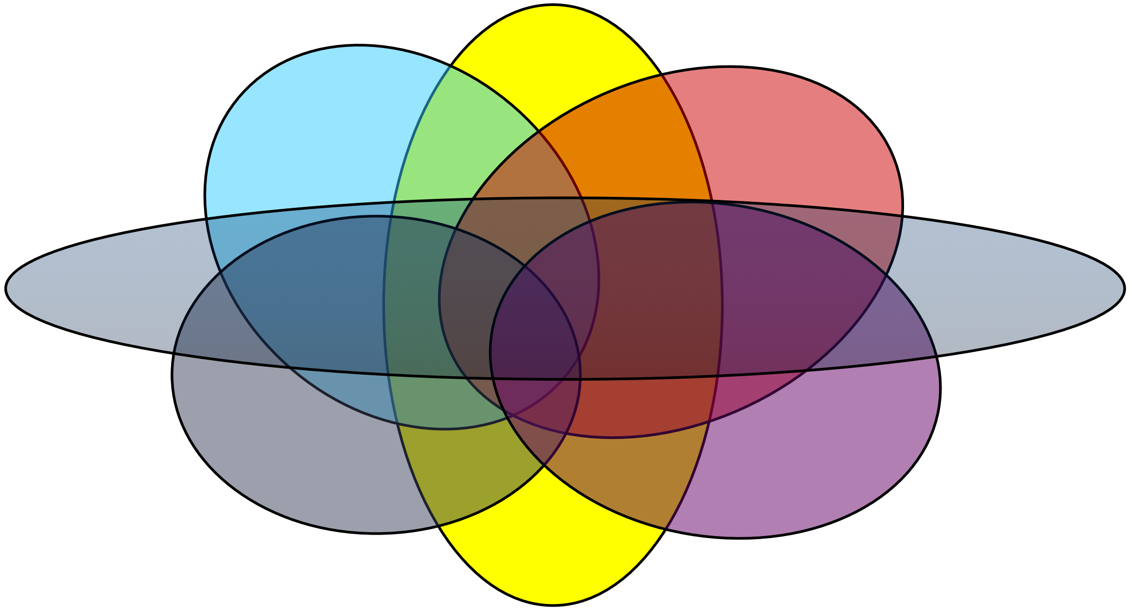
Inclusion-Exclusion (4 Sets)

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$
$$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$$
$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$
$$- |A \cap B \cap C \cap D|$$



Inclusion-Exclusion (n Sets)

What is the inclusion-exclusion formula for the union of n sets?



Inclusion-Exclusion (n Sets)

$$\left|A_1 \cup A_2 \cup \quad \cup A_n\right| =$$

sum of sizes of all single sets
– sum of sizes of all 2-set intersections
+ sum of sizes of all 3-set intersections
– sum of sizes of all 4-set intersections
…
+ $(-1)^{n+1} \times$ sum of sizes of intersections of all n sets

$$= \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{S \subseteq \{1,2,\quad,n\} \\ |S|=k}} \left|_{i \in S} \quad A_i\right|$$

Next class

- Topic: Inclusion-exclusion Principle
- Pre-class reading: Chap 6.4