Lecture 28: Discrete Probabilities

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Outline

- Random variables
- Probability functions
- Continuous cases

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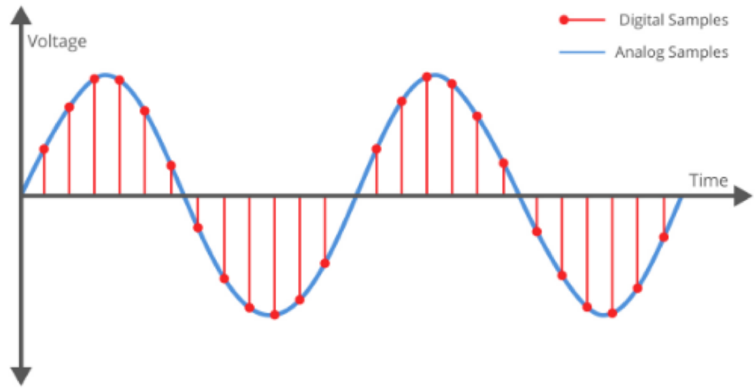
- **Random variables**
- Probability functions
- Continuous cases

Discrete Random Variables

- A Random Variable is a measurement on an outcome of a random experiment.
- Discrete versus Continuous random variable: a random variable x is discrete if it can assume a finite or countably infinite number of values. x is continuous if it can assume all values in an interval.



Example

- Which of the following random variables are discrete and which are continuous?
  - x = Number of houses sold by real estate developer per week?
  - x = Number of heads in ten tosses of a coin?
  - x = Weight of a child at birth?
  - x = Time required to run100 yards?

More examples

- If you roll a die, the outcome is random (not fixed) and there are 6 possible outcomes, each of which occur with probability one-sixth.
- If you poll people about their voting preferences, the percentage of the sample that responds "Yes on Proposition 100" is a also a random variable (the percentage will be slightly differently every time you poll).

Roughly, probability is how frequently we expect different outcomes to occur if we repeat the experiment over and over ("frequentist" view)

Random variables can be discrete or continuous

- **Discrete** random variables have a countable number of outcomes
  - Examples: Dead/alive, treatment/placebo, dice, counts, etc.
- **Continuous** random variables have an infinite continuum of possible values.
  - Examples: blood pressure, weight, the speed of a car, the real numbers from 1 to 6.

Outline

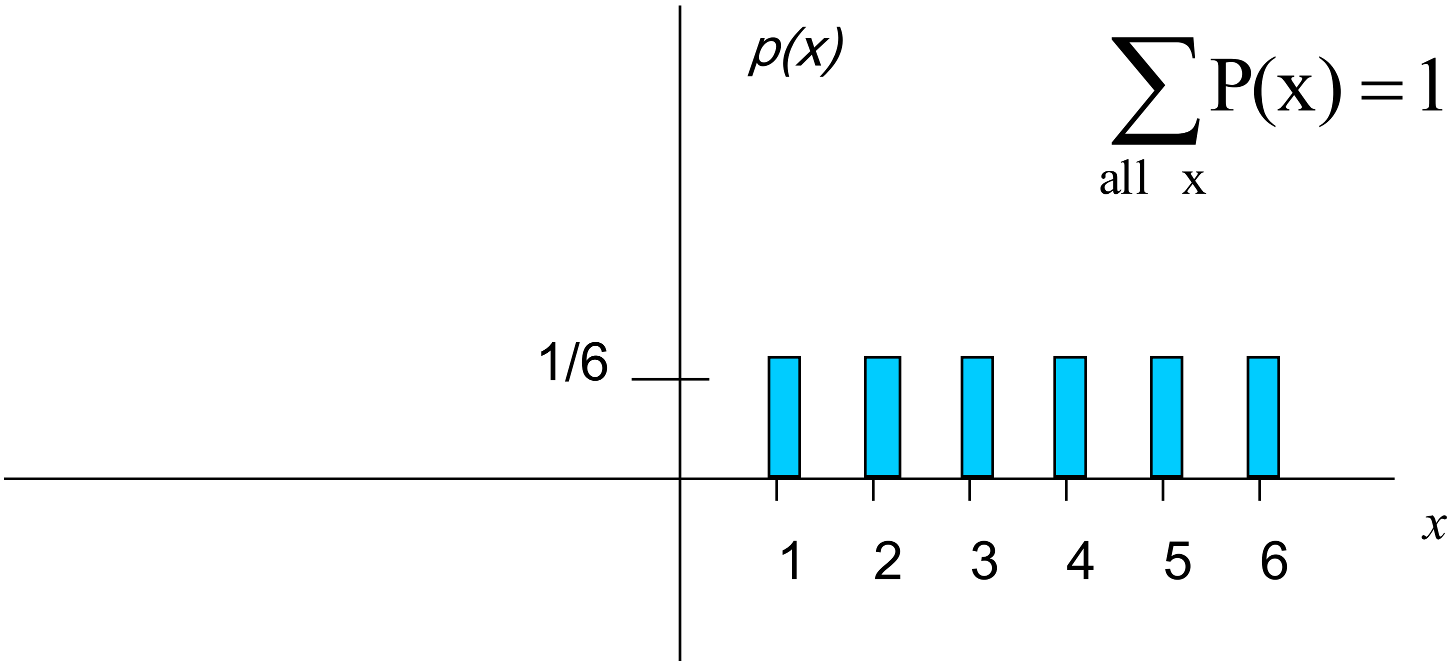
- Random variables
- **Probability functions**
- Continuous cases

Probability functions

If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the *probability* of E is $p(E) = \frac{|E|}{|S|}$.

- A probability function maps the possible values of *x* against their respective probabilities of occurrence, *p(x)*
- *p(x)* is a number from 0 to 1.0.
- The area under a probability function is always 1.

Discrete example: roll of a die



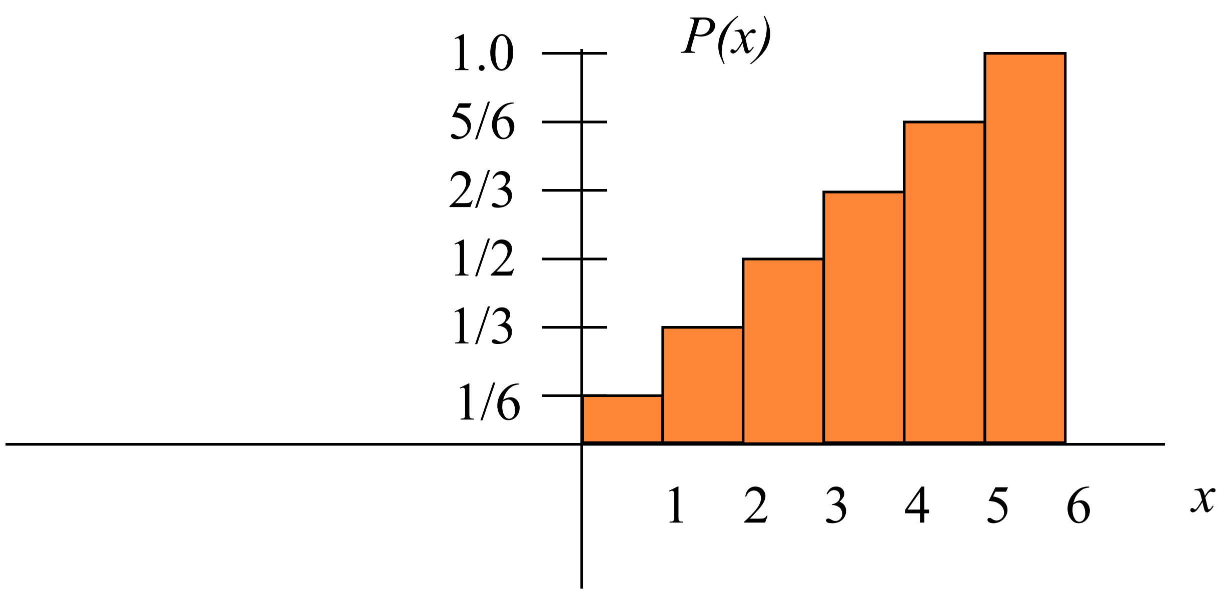
$$\sum_{\text{all } x} P(x) = 1$$

Probability mass function (pmf)

x	$p(x)$
1	$p(x=1)=1/6$
2	$p(x=2)=1/6$
3	$p(x=3)=1/6$
4	$p(x=4)=1/6$
5	$p(x=5)=1/6$
6	$p(x=6)=1/6$

1.0

Cumulative distribution function (CDF)



Cumulative distribution function

x	$P(x \leq A)$
1	$P(x \leq 1) = 1/6$
2	$P(x \leq 2) = 2/6$
3	$P(x \leq 3) = 3/6$
4	$P(x \leq 4) = 4/6$
5	$P(x \leq 5) = 5/6$
6	$P(x \leq 6) = 6/6$

Practice Problem:

- The number of patients seen in the ER in any given hour is a random variable represented by *x*. The probability distribution for *x* is:

x	10	11	12	13	14
P(x)	.4	.2	.2	.1	.1

Find the probability that in a given hour:

a. exactly 14 patients arrive $p(x=14) = .1$

b. At least 12 patients arrive $p(x \geq 12) = (.2 + .1 + .1) = .4$

c. At most 11 patients arrive $p(x \leq 11) = (.4 + .2) = .6$

Example 1

If you toss a die, what's the probability that you roll a 3 or less?

The possible value of the die? **1, 2, 3**, 4, 5, 6

One of these three ≤ 3.

∴ 1/2

Example 2

Two dice are rolled and the sum of the face values is six? What is the probability that at least one of the dice came up a 3?

How can you get a 6 on two dice? 1-5, 5-1, 2-4, 4-2, 3-3

One of these five has a 3.

$\therefore$ 1/5

Property of probability

- We can use counting techniques to find the probability of events derived from other events.

Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

Solution: Let *E* be the event that at least one of the 10 bits is 0. Then $\overline{E}$ is the event that all the bits are 1s. Because the sample space *S* is the set of all bit strings of length 10, it follows that

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

Property of probability

- We can use counting techniques to find the probability of events derived from other events.

Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

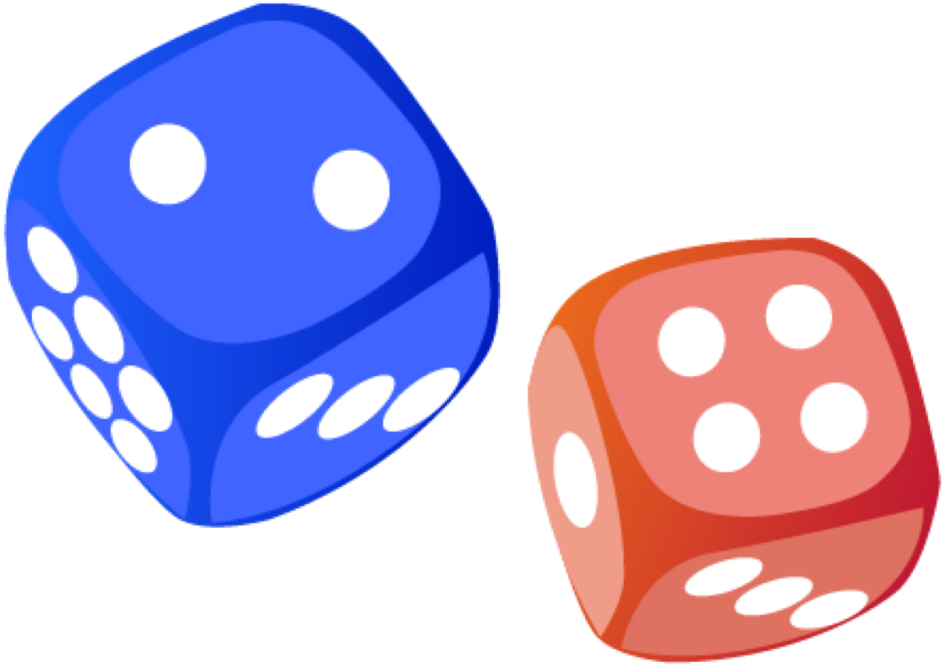
Solution: Let E_1 be the event that the integer selected at random is divisible by 2, and let E_2 be the event that it is divisible by 5.

$\overline{E}$

$$|E_1| = 50, |E_2| = 20, \text{ and } |E_1 \cap E_2| = 10$$

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}.$$

Examples



X is the Sum of Two Dice. What is the probability of X?

Probability Distribution Example: X is the Sum of Two Dice

red green	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

This sequence provides an example of a discrete random variable. Suppose that you have a red die which, when thrown, takes the numbers from 1 to 6 with equal probability.

Probability Distribution Example: X is the Sum of Two Dice

red / green	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Suppose that you also have a green die that can take the numbers from 1 to 6 with equal probability.

Probability Distribution Example: X is the Sum of Two Dice

red / green	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

We will define a random variable X as the sum of the numbers when the dice are thrown.

Probability Distribution Example: X is the Sum of Two Dice

red / green	1	2	3	4	5	6
1						
2						
3						
4						
5						
6				10		

For example, if the red die is 4 and the green one is 6, X is equal to 10.

Probability Distribution Example: X is the Sum of Two Dice

red / green	1	2	3	4	5	6
1						
2						
3						
4						
5		7				
6						

Similarly, if the red die is 2 and the green one is 5, X is equal to 7.

Probability Distribution Example: X is the Sum of Two Dice

red / green	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The table shows all the possible outcomes.

Probability Distribution Example: X is the Sum of Two Dice

red / green	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

X	f
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

We will now define f, the frequencies associated with the possible values of X.

Probability Distribution Example: X is the Sum of Two Dice

green \ red	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

X	f
2	
3	
4	
5	4
6	
7	
8	
9	
10	
11	
12	

For example, there are four outcomes which make X equal to 5.

Probability Distribution Example: X is the Sum of Two Dice

red / green	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

X	f
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

Similarly you can work out the frequencies for all the other values of X.

Probability Distribution Example: X is the Sum of Two Dice

green \ red	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

X	*f*	*p*
2	1	
3	2	
4	3	
5	4	
6	5	
7	6	
8	5	
9	4	
10	3	
11	2	
12	1	

Finally we will derive the probability of obtaining each value of X.

Probability Distribution Example: X is the Sum of Two Dice

green \ red	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

X	f	p
2	1	
3	2	
4	3	
5	4	
6	5	
7	6	
8	5	
9	4	
10	3	
11	2	
12	1	

If there is 1/6 probability of obtaining each number on the red die, and the same on the green die, each outcome in the table will occur with 1/36 probability.

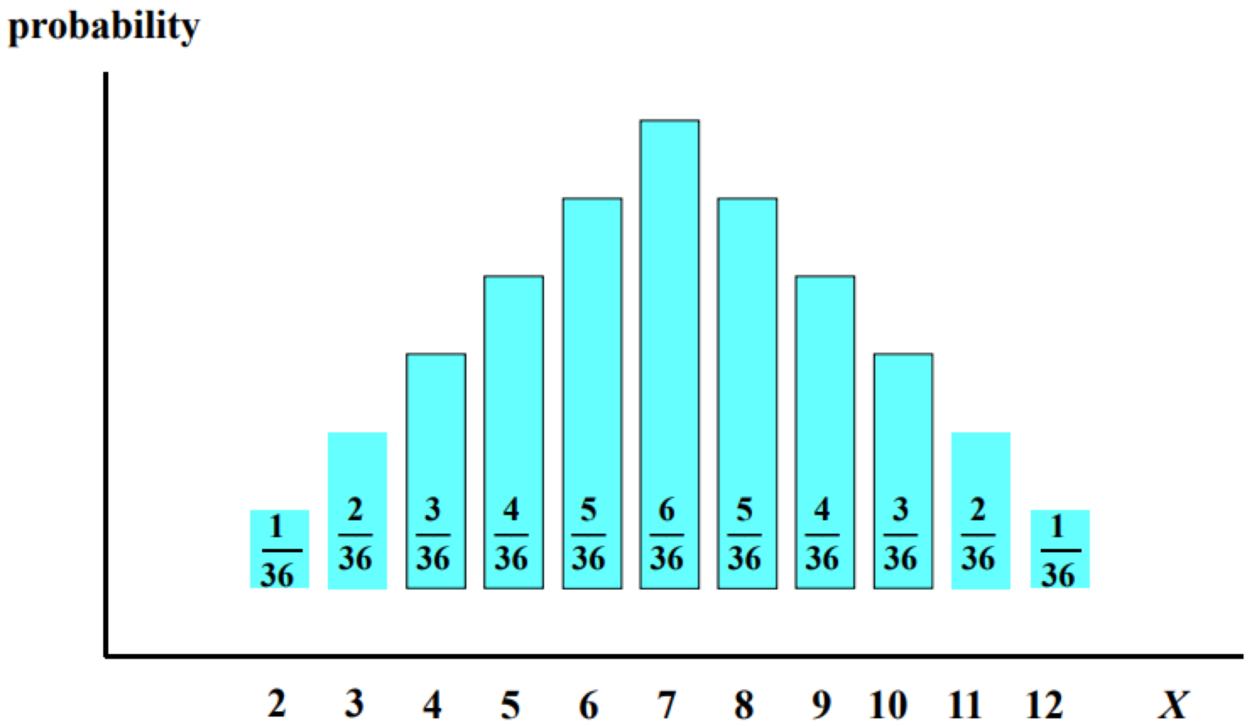
Probability Distribution Example: X is the Sum of Two Dice

red / green	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

X	f	p
2	1	1/36
3	2	2/36
4	3	3/36
5	4	4/36
6	5	5/36
7	6	6/36
8	5	5/36
9	4	4/36
10	3	3/36
11	2	2/36
12	1	1/36

Hence to obtain the probabilities associated with the different values of X, we divide the frequencies by 36.

Probability Distribution Example: X is the Sum of Two Dice



The distribution is shown graphically. in this example it is symmetrical, highest for X equal to 7 and declining on either side.

Outline

- Random variables
- Probability functions
- **Continuous cases**

Continuous case

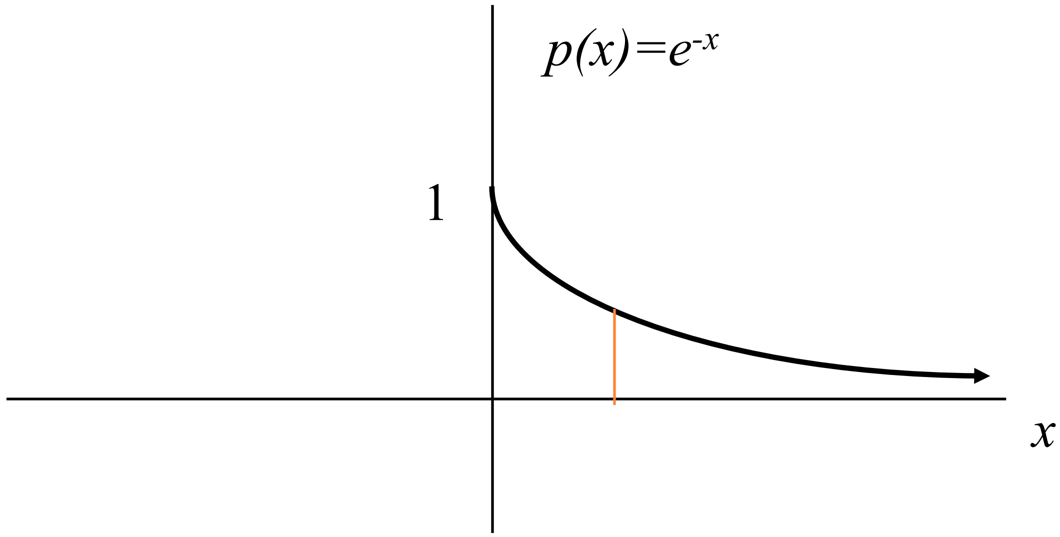
- The probability function that accompanies a continuous random variable is a continuous mathematical function that integrates to 1.
  - For example, recall the negative exponential function (in probability, this is called an "exponential distribution"):

$$f(x) = e^{-x}$$

- This function integrates to 1:

$$\int_0^{+\infty} e^{-x} = -e^{-x} \Big|_0^{+\infty} = 0 + 1 = 1$$

Probability density function for continuous case



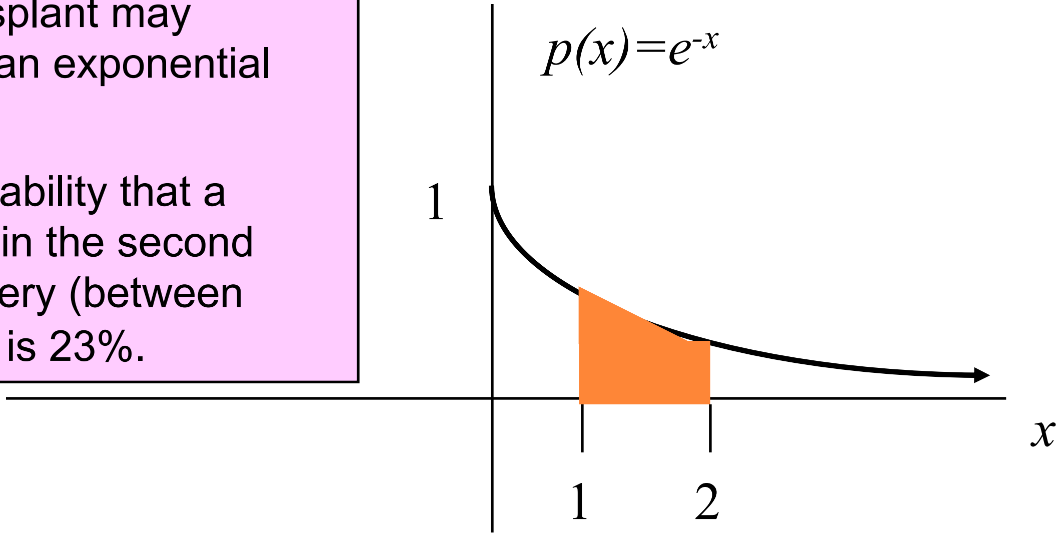
The probability that x is any exact particular value (such as 1.9976) is 0; we can only assign probabilities to possible ranges of x.

Probability density function for continuous case

For example, the probability of *x* falling within 1 to 2:

Clinical example: Survival times after lung transplant may roughly follow an exponential function.

Then, the probability that a patient will die in the second year after surgery (between years 1 and 2) is 23%.

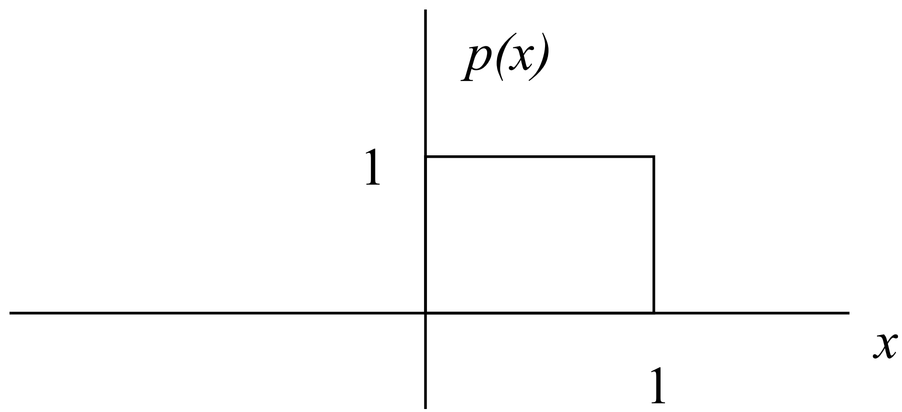


$$P(1 \le x \le 2) = \int_1^2 e^{-x} = -e^{-x} \Big|_1^2 = -e^{-2} - -e^{-1} = -.135 + .368 = .23$$

Example: Uniform distribution

The uniform distribution: all values are equally likely.

$f(x) = 1$, for $1 \geq x \geq 0$

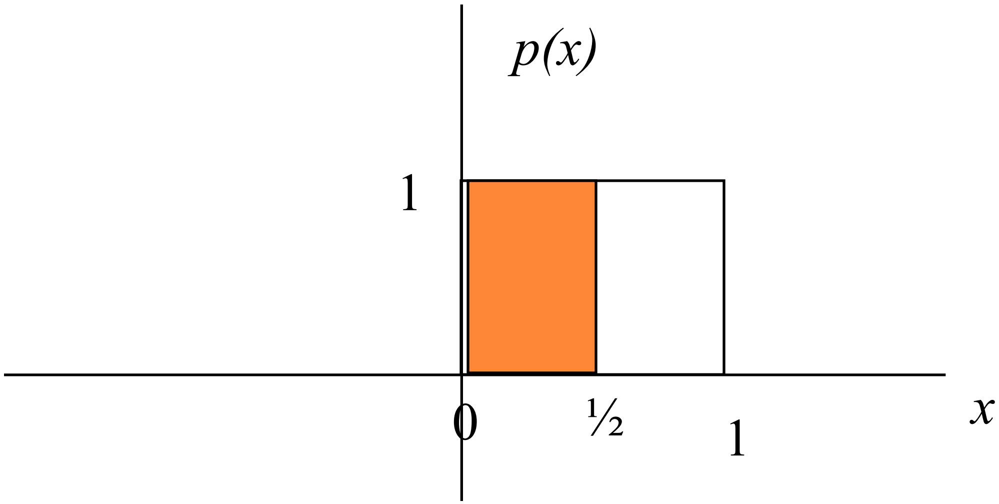


We can see it's a probability distribution because it integrates to 1 (the area under the curve is 1):

$$\int_0^1 1 = x \Big|_0^1 = 1 - 0 = 1$$

Example: Uniform distribution

What's the probability that x is between 0 and ½?



Clinical Research Example: When randomizing patients in an RCT, we often use a random number generator on the computer. These programs work by randomly generating a number between 0 and 1 (with equal probability of every number in between). Then a subject who gets X<.5 is control and a subject who gets X>.5 is treatment.

$P(\frac{1}{2} \geq x \geq 0) = \frac{1}{2}$

Next class

- Topic: Conditional Probability and Bayes Rule
- Pre-class reading: Chap 7.2-7.3