UNIVERSITY AT ALBANY

State University of New York

Lecture 29: Conditional Probabilities and Bayes Rule

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Outline

- Random variables and statistical independence
- Conditional Probability
- Sum and Product Rules
- Bayes Rule

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- **Random variables and statistical independence**
- Conditional Probability
- Sum and Product Rules
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Pairs of Discrete Random Variables

- Let x and y be two discrete r.v.
- For each possible pair of values, we can define a joint probability ***P(x, y)***
- We can also define a joint probability mass function ***P(x,y)*** which offers a complete characterization of the pair of

$$P_x(x) = \sum_{y \in Y} P(x, y)$$

$$P_y(y) = \sum_{x \in X} P(x, y)$$

Marginal distributions

Statistical Independence

- Two random variables x and y are said to be independent, if and only if

$$P(x,y)=P_x(x)\ P_y(y)$$

that is, when knowing the value of x does not give us additional information for the value of y.

- Or, equivalently

$$E[f(x)g(y)] = E[f(x)]\ E[g(y)]$$

for any functions f(x) and g(y).

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Conditional Probability

- When two r.v. are not independent, knowing one allows better estimate of the other (e.g. outside temperature, season)

$$\Pr[x = x_i \mid y = y_j] = \frac{\Pr[x = x_i , y = y_j]}{\Pr[y = y_j]}$$

- If independent P(x|y)=P(x)

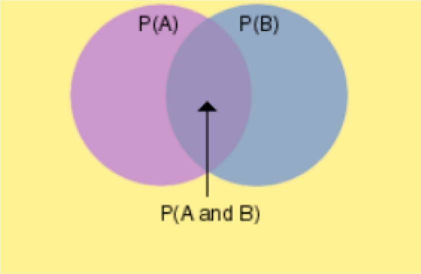
Conditional Probability Example

- A jar contains black and white marbles.
  - Two marbles are chosen without replacement.
  - The probability of selecting a black marble and then a white marble is 0.34.
  - The probability of selecting a black marble on the first draw is 0.47.

- What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

$$P(White \mid Black) = \frac{P(Black \wedge White)}{P(Black)} = \frac{0.34}{0.47} = 0.72$$

A is black in first draw, B is white in second draw



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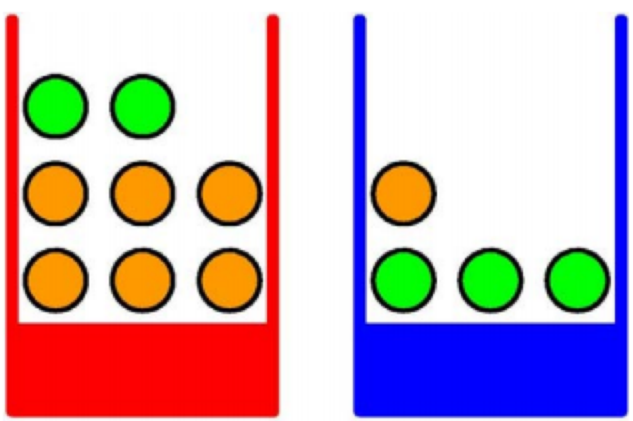
Sum and Product Rules

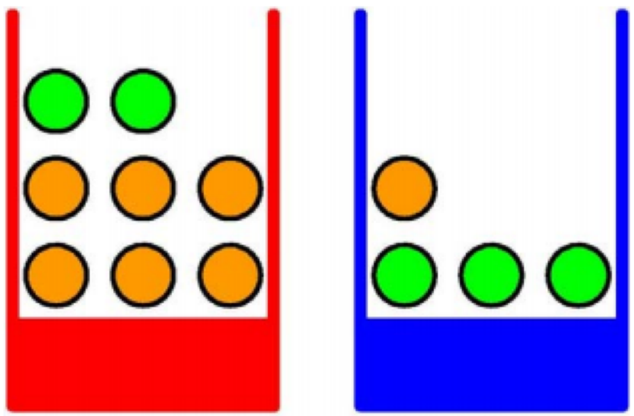
- Example:
  - We have two boxes: one red and one blue
  - Red box: 2 apples and 6 oranges
  - Blue box: 3 apples and 1 orange



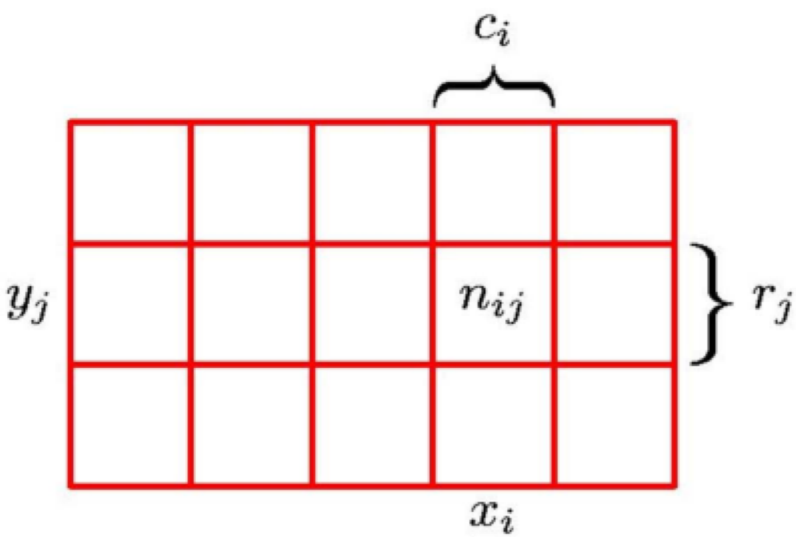
[*C.M. Bishop, "Pattern Recognition and Machine Learning", 2006*]

Sum and Product Rules

- Define:
  - B random variable for box picked (r or b)
  - F identity of fruit (a or o)
- p(B=r)=4/10 and p(B=b)=6/10
  - Events are mutually exclusive and include all possible outcomes ⇨ their probabilities must sum to 1.



Sum and Product Rules



Marginal Probability

$$p(X = x_i) = \frac{c_i}{N}.$$

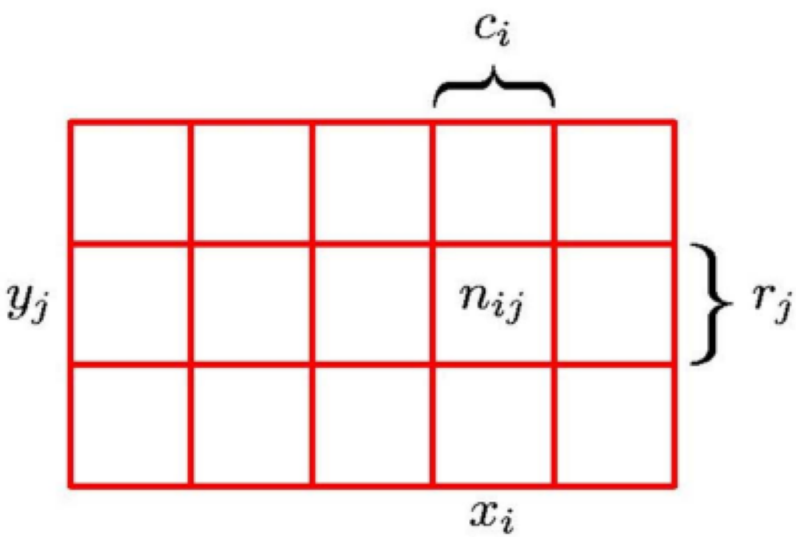
Joint Probability

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

Conditional Probability

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$

Sum and Product Rules



Sum Rule

$$p(X = x_i) = \frac{c_i}{N} = \frac{1}{N}\sum_{j=1}^{L} n_{ij}$$

$$= \sum_{j=1}^{L} p(X = x_i, Y = y_j)$$

Product Rule

$$\begin{aligned} p(X = x_i, Y = y_j) &= \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N} \\ &= p(Y = y_j \mid X = x_i)\, p(X = x_i) \end{aligned}$$

Sum and Product Rules

- Sum Rule $$p(X) = \sum_{Y} p(X, Y)$$
- Product Rule $$p(X, Y) = p(Y|X)p(X)$$

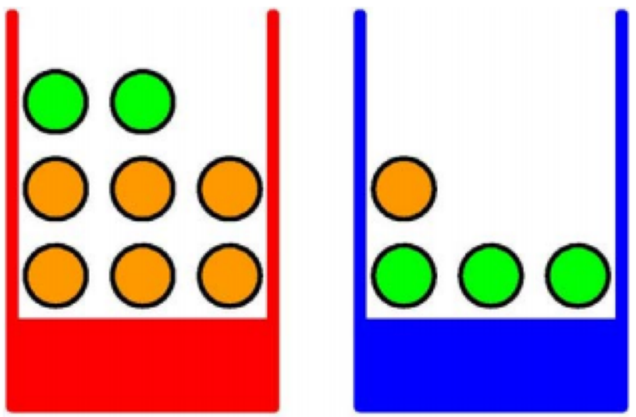
Law of Total Probability

- If an event A can occur in m different ways and if these m different ways are mutually exclusive, then the probability of A occurring is the sum of the probabilities of the sub-events

$$P(X = x_i) = \sum_j P(X = x_i \mid Y = y_j)P(Y = y_j)$$

Sum and Product Rules

- Back to the fruit baskets
  - p(B=r)=4/10 and p(B=b)=6/10
  - p(B=r) + p(B=b) = 1
- Conditional probabilities
  - p(F=a | B = r) = 1/4
  - p(F=o | B = r) = 3/4
  - p(F=a | B = b) = 3/4
  - p(F=o | B = b) = 1/4



Sum and Product Rules

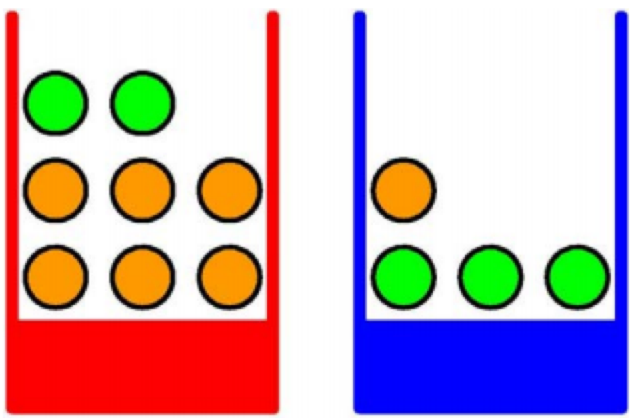
- Note:

$p(F=a \mid B = r) + p(F=o \mid B = r) = 1$

$p(F=a) = p(F=a \mid B = r)\, p(B=r) + p(F=a \mid B = b) p(B=b)$

$= 1/4 * 4/10 + 3/4 * 6/10 = 11/20$

- Sum rule: $p(F=o) = ?$



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- **Bayes Rule**

Law of Total Probability

$$P_x(x) = \sum_{y \in Y} P(x, y)$$

$$P(x \mid y) = \frac{P(x, y)}{P(y)}$$

Bayes Rule

$$P(x \mid y) = \frac{P(x,y)}{P(y)} = \frac{P(y \mid x)P(x)}{\sum_{x \in X} P(x,y)}$$

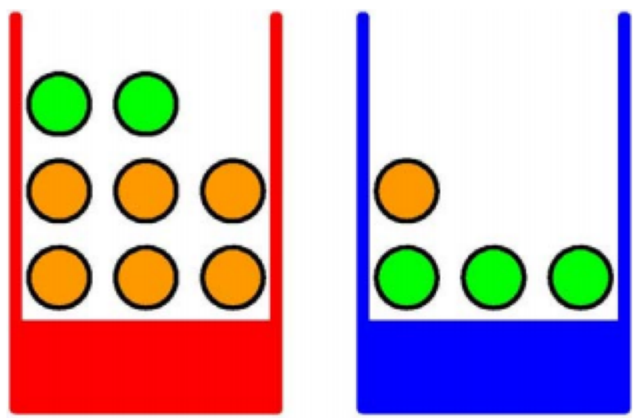
$$\text{posterior} = \frac{\text{likelihood} * \text{prior}}{\text{evidence}}$$

- x is the unknown cause
- y is the observed evidence
- Bayes rule shows how probability of x changes after we have observed y

Bayes Rule on the Fruit Example

- Suppose we have selected an orange. Which box did it come from?

$$p(B=r \mid F=o) = \frac{p(F=o \mid B=r)\,p(B=r)}{p(F=o)} = \frac{\frac{3}{4} \times \frac{4}{10}}{\frac{9}{20}} = \frac{2}{3}$$



Continuous Random Variables

- Examples: room temperature, time to run100m, weight of child at birth…
- Cannot talk about probability of that x has a particular value
- Instead, probability that x falls in an interval ⇨ **probability density function**

$$\Pr[x \in (a,b)] = \int_a^b p(x)dx$$

$$p(x) \geq 0 \text{ and } \int_{-\infty}^{\infty} p(x)dx = 1$$

Bayes Rule for Continuous Case

- Bayes rule $$p(x \mid y) = \frac{p(y \mid x)\, p(x)}{\int_{-\infty}^{\infty} p(y \mid x)\, p(x)\, dx}$$

$$\text{posterior} = \frac{\text{likelihood} * \text{prior}}{\text{evidence}}$$

Next class

- Topic: Expected Value, Variance and Binormal Distribution
- Pre-class reading: Chap 7.4