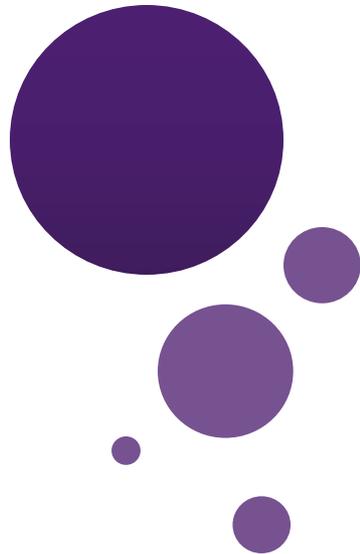




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## Lecture 3: Predicate Logic and Quantifiers



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# Outline

- Logical Equivalences
- Introduction to Predicate Logic
- Quantifiers

# Outline

- **Logical Equivalences**
- Introduction to Predicate Logic
- Quantifiers

# Logical Equivalences: Example 1

- Are the propositions  $(p \rightarrow q)$  and  $(\neg p \vee q)$  logically equivalent?
- To find out, we construct the truth tables for each:

$p$	$q$	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

The two columns in the truth table are identical, thus we conclude that

$$(p \rightarrow q) \equiv (\neg p \vee q)$$

# Logical Equivalences: Example 2

- Show that  $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

$p$	$q$	$r$	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \vee (q \rightarrow r)$	$p \wedge q$	$(p \wedge q) \rightarrow r$
0	0	0	1	1	1	0	1
0	0	1	1	1	1	0	1
0	1	0	1	0	1	0	1
0	1	1	1	1	1	0	1
1	0	0	0	1	1	0	1
1	0	1	1	1	1	0	1
1	1	0	0	0	0	1	0
1	1	1	1	1	1	1	1

# Logical Equivalences: Cheat Sheet (1)

## Logical Equivalences Involving Conditional Statements.

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

# Logical Equivalences: Cheat Sheet (2)

Identities (Equivalences)	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

# Using Logical Equivalences: Example 1

- Logical equivalences can be used to construct additional logical equivalences
- Example: Show that  $(p \wedge q) \rightarrow q$  is a tautology

0.  $(p \wedge q) \rightarrow q$

1.  $\equiv \neg(p \wedge q) \vee q$

Implication Law on 0

2.  $\equiv (\neg p \vee \neg q) \vee q$

De Morgan's Law (1<sup>st</sup>) on 1

3.  $\equiv \neg p \vee (\neg q \vee q)$

Associative Law on 2

4.  $\equiv \neg p \vee 1$

Negation Law on 3

5.  $\equiv 1$

Domination Law on 4

# Using Logical Equivalences: Example 3

- Example (Exercise 17)\*: Show that  $\neg(p \leftrightarrow q) \equiv (p \leftrightarrow \neg q)$
- Sometimes it helps to start with the second proposition

$$(p \leftrightarrow \neg q)$$

0.  $(p \leftrightarrow \neg q)$

1.  $\equiv (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$

2.  $\equiv (\neg p \vee \neg q) \wedge (q \vee p)$

3.  $\equiv \neg(\neg((\neg p \vee \neg q) \wedge (q \vee p)))$

4.  $\equiv \neg(\neg(\neg p \vee \neg q) \vee \neg(q \vee p))$

5.  $\equiv \neg((p \wedge q) \vee (\neg q \wedge \neg p))$

6.  $\equiv \neg((p \vee \neg q) \wedge (p \vee \neg p) \wedge (q \vee \neg q) \wedge (q \vee \neg p))$

7.  $\equiv \neg((p \vee \neg q) \wedge (q \vee \neg p))$

8.  $\equiv \neg((q \rightarrow p) \wedge (p \rightarrow q))$

9.  $\equiv \neg(p \leftrightarrow q)$

Equivalence Law on 0

Implication Law on 1

Double negation on 2

De Morgan's Law...

De Morgan's Law

Distribution Law

Identity Law

Implication Law

Equivalence Law

\*See Table 8 (p 25) but you are not allowed to use the table for the proof

# Using Logical Equivalences: Example 3

- Show that  $\neg(q \rightarrow p) \vee (p \wedge q) \equiv q$

0.  $\neg(q \rightarrow p) \vee (p \wedge q)$

1.  $\equiv \neg(\neg q \vee p) \vee (p \wedge q)$

2.  $\equiv (q \wedge \neg p) \vee (p \wedge q)$

3.  $\equiv (q \wedge \neg p) \vee (q \wedge p)$

4.  $\equiv q \wedge (\neg p \vee p)$

5.  $\equiv q \wedge 1$

$\equiv q$

Implication Law

De Morgan's & Double  
negation

Commutative Law

Distributive Law

Identity Law

Identity Law

# Logic in Programming: Example 2 (revisited)

- Recall the loop

```
While
```

```
((i<size AND A[i]>10) OR  
(i<size AND A[i]<0) OR  
(i<size AND (NOT (A[i]!=0 AND NOT (A[i]>=10))))))
```

- Now, using logical equivalences, simplify it!
- Using De Morgan's Law and Distributivity

```
While ((i<size) AND  
      ((A[i]>10 OR A[i]<0) OR  
      (A[i]==0 OR A[i]>=10)))
```

- Noticing the ranges of the 4 conditions of  $A[i]$

```
While ((i<size) AND (A[i]>=10 OR A[i]<=0))
```

# Programming Pitfall Note

- In C, C++ and Java, applying the commutative law is not such a good idea.
- For example, consider accessing an integer array A of size n:

```
if (i < n && A[i] == 0) i++;
```

is not equivalent to

```
if (A[i] == 0 && i < n) i++;
```

# Outline

- Logical Equivalences
- **Introduction to Predicate Logic**
- Quantifiers

# Introduction

- Consider the statements:

$$x > 3, x = y + 3, x + y = z$$

- The symbols  $>$ ,  $+$ ,  $=$  denote relations between  $x$  and  $3$ ,  $x$ ,  $y$ , and  $4$ , and  $x, y$ , and  $z$ , respectively
- These relations may hold or not hold depending on the values that  $x$ ,  $y$ , and  $z$  may take.
- A **predicate** is a property that is affirmed or denied about the subject (in logic, we say '**variable**' or '**argument**') of a statement
- Consider the statement : 'x is greater than 3'
  - 'x' is the subject
  - 'is greater than 3' is the predicate

# Propositional Functions (1)

- To write in Predicate Logic ‘ $x$  is greater than 3’
  - We introduce a functional symbol for the **predicate** and
  - Put the subject as an **argument** (to the functional symbol):  $P(x)$
- Terminology
  - $P(x)$  is a statement
  - $P$  is a predicate or propositional function
  - $x$  as an argument

# Propositional Functions (2)

- **Examples:**
  - $\text{Father}(x)$ : unary predicate
  - $\text{Brother}(x,y)$ : binary predicate
  - $\text{Sum}(x,y,z)$ : ternary predicate
  - $P(x,y,z,t)$ : n-ary predicate

# Propositional Functions (3)

- **Definition:** A statement of the form  $P(x_1, x_2, \dots, x_n)$  is the value of the propositional symbol  $P$ .
- Here:  $(x_1, x_2, \dots, x_n)$  is an  $n$ -tuple and  $P$  is a predicate
- We can think of a propositional function as a function that
  - Evaluates to true or false
  - Takes one or more arguments
  - Expresses a predicate involving the argument(s)
  - Becomes a proposition when values are assigned to the arguments

# Propositional Functions: Example

- Let  $Q(x,y,z)$  denote the statement ' $x^2+y^2=z^2$ '
  - What is the truth value of  $Q(3,4,5)$ ?  
 $Q(3,4,5)$  is true
  - What is the truth value of  $Q(2,2,3)$ ?  
 $Q(2,3,3)$  is false
  - How many values of  $(x,y,z)$  make the predicate true?  
There are infinitely many values that make the proposition true, how many right triangles are there?

# Universe of Discourse

- Consider the statement ' $x > 3$ ', does it make sense to assign to  $x$  the value 'blue'?
- Intuitively, the **universe of discourse** is the set of all things we wish to talk about; that is the set of all objects that we can sensibly assign to a variable in a propositional function.
- What would be the universe of discourse for the propositional function below be:

Enrolled ICSI210( $x$ ) = 'x is enrolled in ICSI 210'

# Universe of Discourse: Multivariate functions

- Each variable in an  $n$ -tuple (i.e., each argument) may have a different universe of discourse
- Consider an  $n$ -ary predicate  $P$ :  
 $P(r,g,b,c)$  = 'The  $rgb$ -values of the color  $c$  is  $(r,g,b)$ '
- Example, what is the truth value of
  - $P(255,0,0,red)$
  - $P(0,0,255,green)$
- What are the universes of discourse of  $(r,g,b,c)$ ?

# Outline

- Logical Equivalences
- Introduction to Predicate Logic
- **Quantifiers**

# Quantifiers: Introduction

- The statement ' $x > 3$ ' is not a proposition
- It becomes a proposition
  - When we assign values to the argument: ' $4 > 3$ ' is false, ' $2 < 3$ ' is true, or
  - When we quantify the statement
- Two quantifiers
  - Universal quantifier  $\forall$   
 $\$forall\$$   
the proposition is true for **all** possible values in the universe of discourse
  - Existential quantifier  $\exists$   
 $\$exists\$$   
the proposition is true for **some** value(s) in the universe of discourse

# Universal Quantifier: Definition

- **Definition:** The universal quantification of a predicate  $P(x)$  is the proposition ' $P(x)$  is true for all values of  $x$  in the universe of discourse.'

We use the notation:  $\forall x P(x)$ , which is read 'for all  $x$ '.

- If the universe of discourse is finite, say  $\{n_1, n_2, \dots, n_k\}$ , then the universal quantifier is simply the conjunction of the propositions over all the elements

$$\forall x P(x) \Leftrightarrow P(n_1) \wedge P(n_2) \wedge \dots \wedge P(n_k)$$

# Universal Quantifier: Example 1

- Let  $P(x)$ : ‘ $x$  must take a discrete mathematics course’ and  $Q(x)$ : ‘ $x$  is a CS student.’
- The universe of discourse for both  $P(x)$  and  $Q(x)$  is all UNL students.
- Express the statements:
  - “Every CS student must take a discrete mathematics course.”  
$$\forall x Q(x) \rightarrow P(x)$$
  - “Everybody must take a discrete mathematics course or be a CS student.”  
$$\forall x ( P(x) \vee Q(x) )$$
  - “Everybody must take a discrete mathematics course and be a CS student.”  
$$\forall x ( P(x) \wedge Q(x) )$$

Are these statements true or false?

# Universal Quantifier: Example 2

- Express the statement: 'for every  $x$  and every  $y$ ,  $x+y>10$ '
- Answer:
  - Let  $P(x,y)$  be the statement  $x+y>10$
  - Where the universe of discourse for  $x, y$  is the set of integers
  - The statement is:  $\forall x \forall y P(x,y)$
- Shorthand:  $\forall x,y P(x,y)$

# Existential Quantifier: Definition

- **Definition:** The existential quantification of a predicate  $P(x)$  is the proposition 'There exists a value  $x$  in the universe of discourse such that  $P(x)$  is true.' We use the notation:  $\exists x P(x)$ , which is read 'there exists  $x$ '.
- If the universe of discourse is finite, say  $\{n_1, n_2, \dots, n_k\}$ , then the existential quantifier is simply the disjunction of the propositions over all the elements

$$\exists x P(x) \Leftrightarrow P(n_1) \vee P(n_2) \vee \dots \vee P(n_k)$$

# Existential Quantifier: Example 1

- Let  $P(x,y)$  denote the statement 'x+y=5'
- What does the expression  $\exists x \exists y P(x,y)$  mean?
- Which universe(s) of discourse make it true?

# Existential Quantifier: Example 2

- Express the statement: ‘there exists a real solution to  $ax^2+bx-c=0$ ’
- Answer:
  - Let  $P(x)$  be the statement  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  - Where the universe of discourse for  $x$  is the set of real numbers. Note here that  $a, b, c$  are fixed constants.
  - The statement can be expressed as  $\exists x P(x)$
- What is the truth value of  $\exists x P(x)$ ?
  - It is false. When  $b^2 < 4ac$ , there are no real number  $x$  that can satisfy the predicate
- What can we do so that  $\exists x P(x)$  is true?
  - Change the universe of discourse to the complex numbers,  $\mathbb{C}$

# Quantifiers: Truth values

- In general, when are quantified statements true or false?

Statement	True when...	False when...
$\forall x P(x)$	$P(x)$ is true for every $x$	There is an $x$ for which $P(x)$ is false
$\exists x P(x)$	There is an $x$ for which $P(x)$ is true	$P(x)$ is false for every $x$

# Mixing quantifiers (1)

- Existential and universal quantifiers can be used together to quantify a propositional predicate. For example:

$$\forall x \exists y P(x,y)$$

is perfectly valid

- Alert:
  - The quantifiers must be read from left to right
  - The order of the quantifiers is important
  - $\forall x \exists y P(x,y)$  is not equivalent to  $\exists y \forall x P(x,y)$

## Mixing quantifiers (2)

- Consider
  - $\forall x \exists y \text{ Loves}(x,y)$ : Everybody loves somebody
  - $\exists y \forall x \text{ Loves}(x,y)$ : There is someone loved by everyone
- The two expressions do not mean the same thing
- $(\exists y \forall x \text{ Loves}(x,y)) \rightarrow (\forall x \exists y \text{ Loves}(x,y))$  but the converse does not hold
- However, you can commute similar quantifiers
  - $\forall x \forall y P(x,y)$  is equivalent to  $\forall y \forall x P(x,y)$  (thus,  $\forall x,y P(x,y)$ )
  - $\exists x \exists y P(x,y)$  is equivalent to  $\exists y \exists x P(x,y)$  (thus  $\exists x,y P(x,y)$ )

# Mixing Quantifiers: Truth values

Statement	True when...	False when...
$\forall x \forall y P(x,y)$	$P(x,y)$ is true for every pair $x,y$	There is at least one <i>pair</i> $x,y$ for which $P(x,y)$ is false
$\forall x \exists y P(x,y)$	For every $x$ , there is a $y$ for which $P(x,y)$ is true	There is an $x$ for which $P(x,y)$ is false for every $y$
$\exists x \forall y P(x,y)$	There is an $x$ for which $P(x,y)$ is true for every $y$	For every $x$ , there is a $y$ for which $P(x,y)$ is false
$\exists x \exists y P(x,y)$	There is at least one pair $x,y$ for which $P(x,y)$ is true	$P(x,y)$ is false for every pair $x,y$

# Next class

- Topic: Quantifiers and Rules of Reference
- Pre-class reading: Chap 1.4-1.5

