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Lecture 33: Graph Theory (1)

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Outline

- Graph and Graph Models
- Graph Terminology and Special Types of Graphs
- Representing Graphs and Graph Isomorphism

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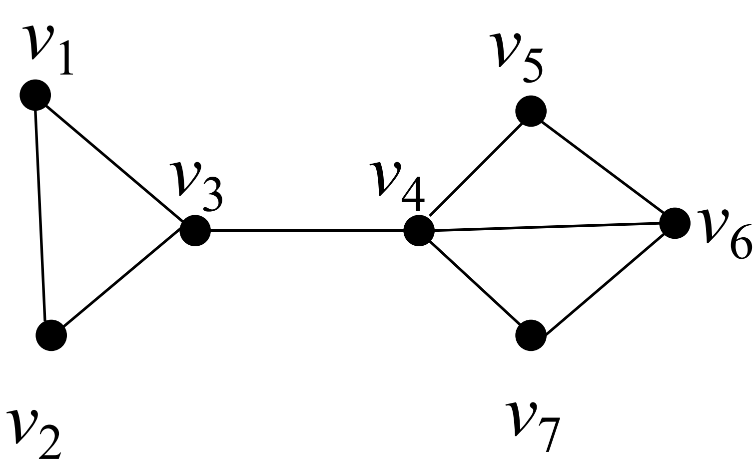
- **Graph and Graph Models**
- Graph Terminology and Special Types of Graphs
- Representing Graphs and Graph Isomorphism

Introduction to Graphs

Def 1. A graph ***G* = (*V*, *E*)** consists of ***V***, a nonempty set of vertices (or nodes), and ***E***, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.



$G=(V, E)$, where

$V=\{v_1, v_2, \ldots, v_7\}$

$E=\{\{v_1,v_2\}, \{v_1,v_3\}, \{v_2,v_3\}, \{v_3,v_4\}, \{v_4,v_5\}, \{v_4,v_6\}, \{v_4,v_7\}, \{v_5,v_6\}, \{v_6,v_7\}\}$

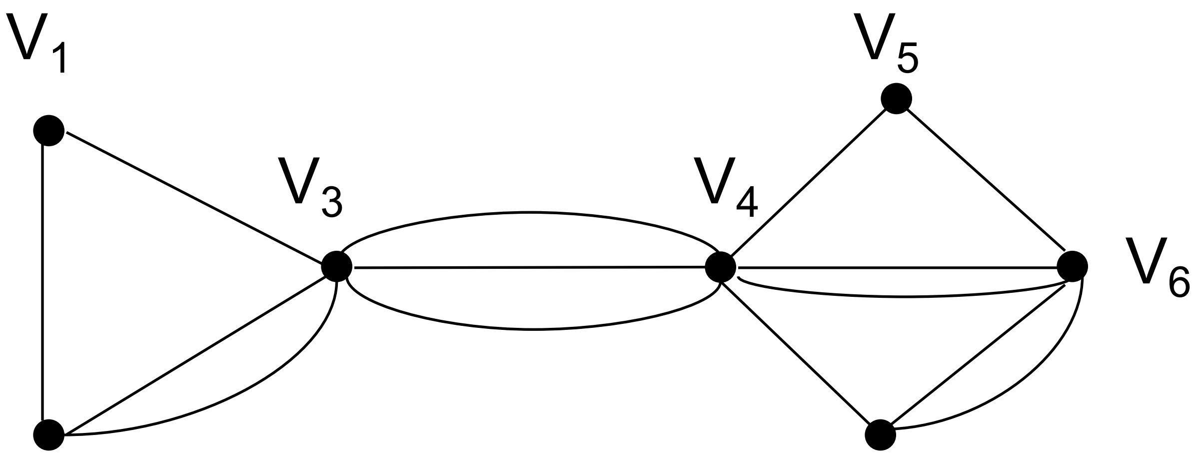
Introduction to Graphs

- A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Multigraph:

simple graph + multiple edges (multiedges)

(Between two points to allow multiple edges)

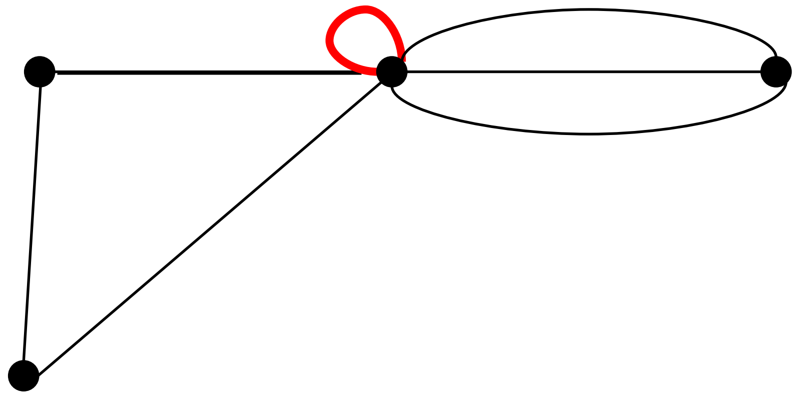


Introduction to Graphs

Def. Pseudograph:

simple graph + multiedge
+ loop

(a loop: )

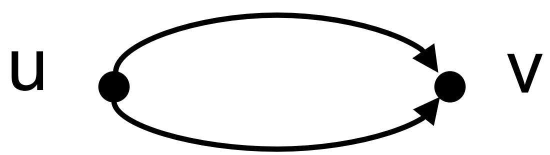


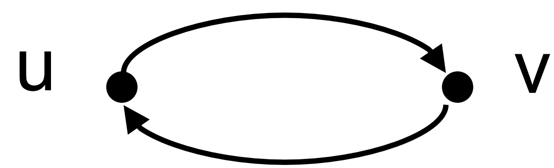
Introduction to Graphs

Def 2. Directed graph (digraph):

simple graph with each edge directed

Note:  is allowed in a directed graph





The two edges (u,v),(u,v) are multiedges.

The two edges (u,v), (v,u) are not multiedges.

Def. Directed multigraph: digraph+multiedges

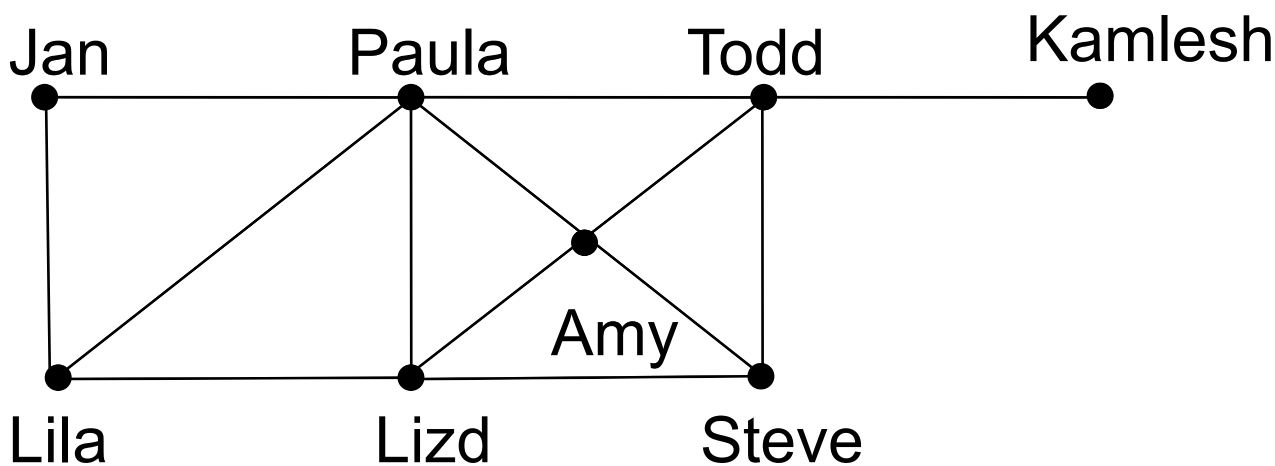
Introduction to Graphs

Table 1. Graph Terminology

Type	Edges	Multiple Edges	Loops
(simple) graph	undirected edge: {u,v}	×	×
Multigraph		✓	×
Pseudograph		✓	✓
Directed graph	directed edge: (u,v)	×	✓
Directed multigraph		✓	✓

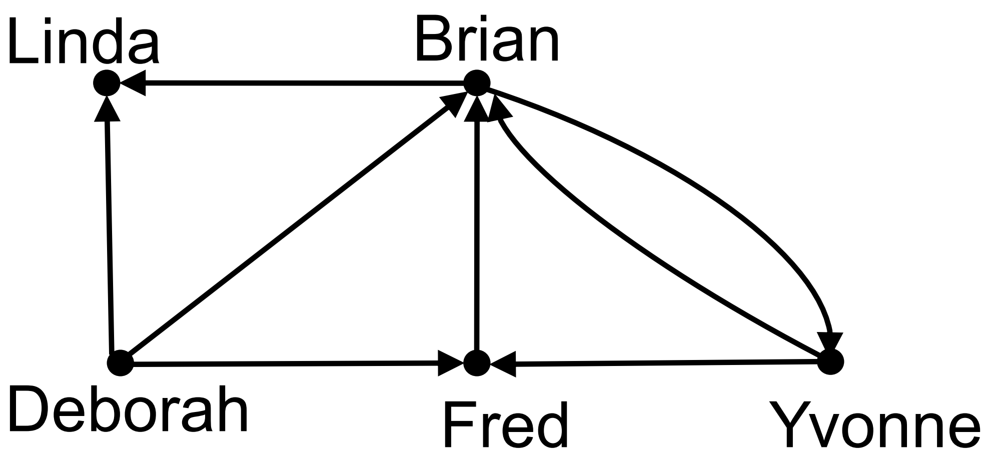
Graph Models: Acquaintance graph

- We can use a simple graph to represent whether two people know each other. Each person is represented by a vertex. An undirected edge is used to connect two people when these people know each other.



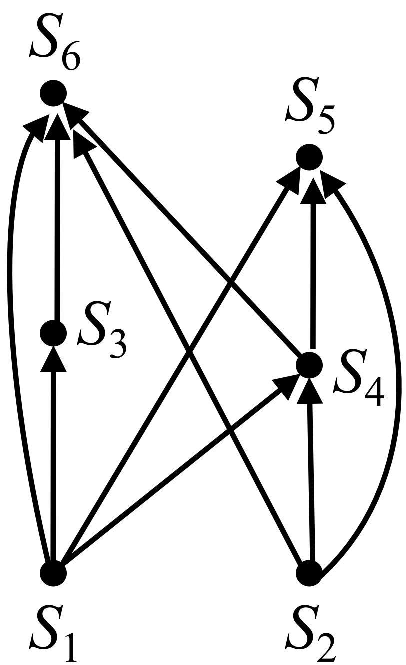
Graph Models: Influence graph

- In studies of group behavior it is observed that certain people can influence the thinking of others.
  Simple digraph $\Rightarrow$ Each person of the group is represented by a vertex. There is a directed edge from vertex a to vertex b when the person a influences the person b.



Graph Models: Precedence graph and concurrent processing

- Computer programs can be executed more rapidly by executing certain statements concurrently. It is important not to execute a statement that requires results of statements not yet executed.
- Simple digraph ⇒ Each statement is represented by a vertex, and there is an edge from a to b if the statement of b cannot be executed before the statement of a.



S_1: a:=0
S_2: b:=1
S_3: c:=a+1
S_4: d:=b+a
S_5: e:=d+1
S_6: e:=c+d

Outline

- Graph and Graph Models
- **Graph Terminology and Special Types of Graphs**
- Representing Graphs and Graph Isomorphism

Graph Terminology

Def 1. Two vertices u and v in a undirected graph G are called adjacent (or neighbors) in G if $\{u, v\}$ is an edge of G.

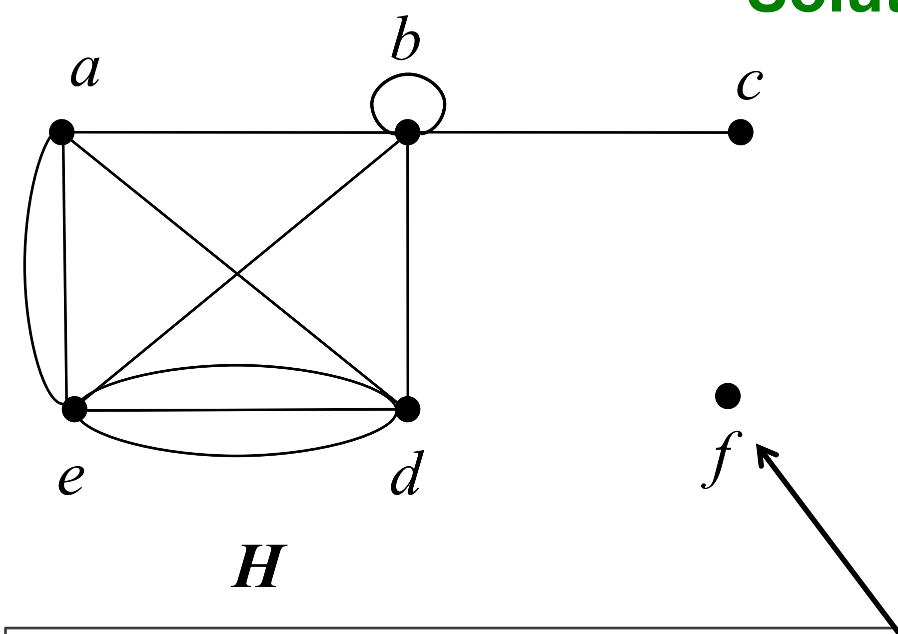
Note : adjacent: a vertex connected to a vertex
incident: a vertex connected to an edge

Def 2. The degree of a vertex v, denoted by deg(v), in an undirected graph is the number of edges incident with it.

(Note : A loop adds 2 to the degree.)

Example

- What are the degrees of the vertices in the graph $\boldsymbol{H}$?



Solution :

$\deg(a)=4$

$\deg(b)=6$

$\deg(c)=1$

$\deg(d)=5$

$\deg(e)=6$

$\deg(f)=0$

Def. A vertex of degree 0 is called isolated.

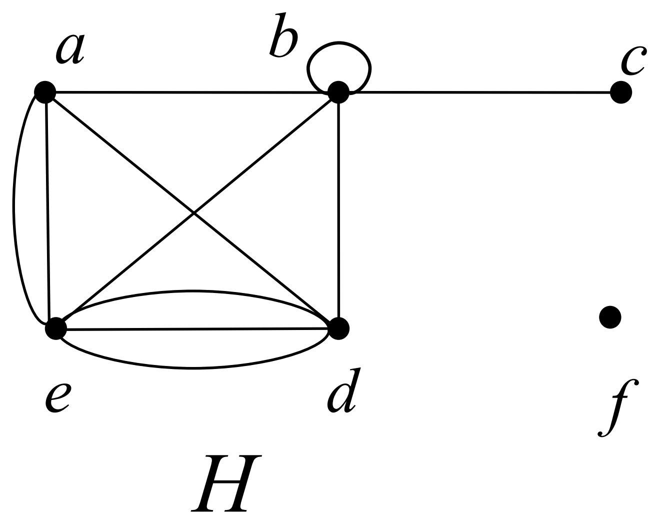
Def. A vertex is **pendant if and only if** it has degree one.

The Handshaking Theorem

- Let $G = (V, E)$ be an undirected graph with e edges (i.e., $|E| = e$). Then

$$\sum_{v \in V} \deg(v) = 2e$$

Example: The graph H has 11 edges, and



$$\sum_{v \in V} \deg(v) = 22$$

Even number of odd degree

- An undirected graph $G = (V, E)$ has an even number of vertices of odd degree.

Proof : Let $V_1 = \{v \in V \mid \deg(v) \text{ is even}\}$,
$V_2 = \{v \in V \mid \deg(v) \text{ is odd}\}$.

$$2e = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

$$\Rightarrow \sum_{v \in V_2} \deg(v) \text{ is even.}$$

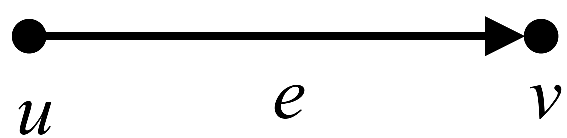
Directed Graph

Def 3. $G = (V, E)$: directed graph,

$e = (u, v) \in E$: u is adjacent to v

v is adjacent from u

- u : initial vertex of e
  v : terminal (end) vertex of e



The initial vertex and terminal vertex of a loop are the same

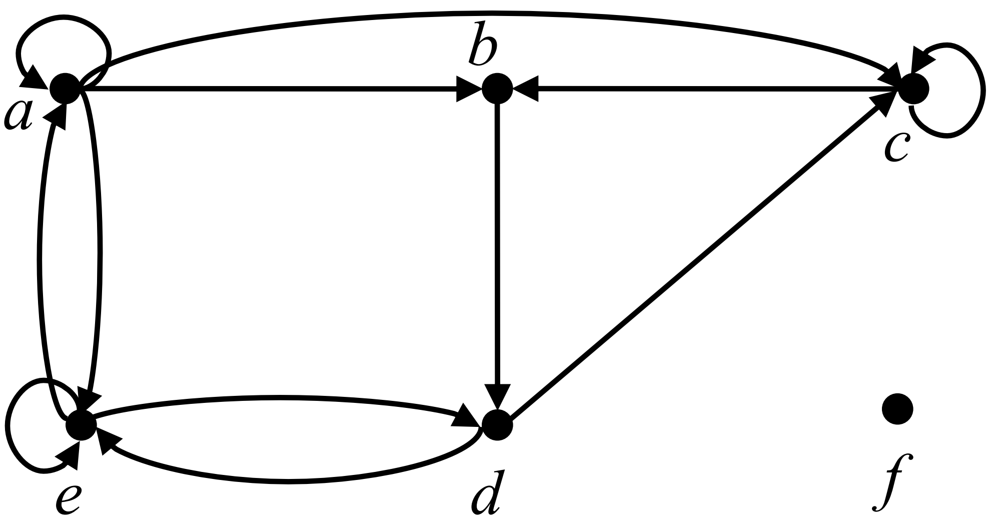
Directed Graph

$G = (V, E)$: directed graph, $v \in V$

$\deg^-(v)$: # of edges with v as a terminal.(in-degree)

$\deg^+(v)$: # of edges with v as a initial vertex. (out-degree)

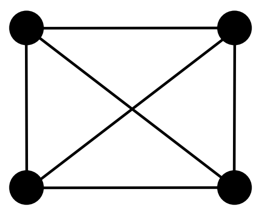
$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$



$\deg^-(a)=2$, $\deg^+(a)=4$
$\deg^-(b)=2$, $\deg^+(b)=1$
$\deg^-(c)=3$, $\deg^+(c)=2$
$\deg^-(d)=2$, $\deg^+(d)=2$
$\deg^-(e)=3$, $\deg^+(e)=3$
$\deg^-(f)=0$, $\deg^+(f)=0$

Regular Graph

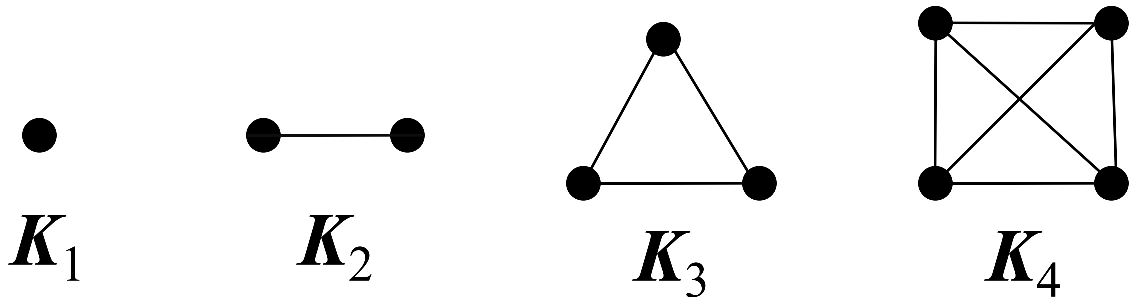
- A simple graph $G=(V, E)$ is called regular if every vertex of this graph has the same degree. A regular graph is called n-regular if $\deg(v)=n$, $\forall v \in V$.



is 3-regular.

Some Special Simple Graphs

- The complete graph on n vertices, denoted by $\boldsymbol{K_n}$, is the simple graph that contains exactly one edge between each pair of distinct vertices.

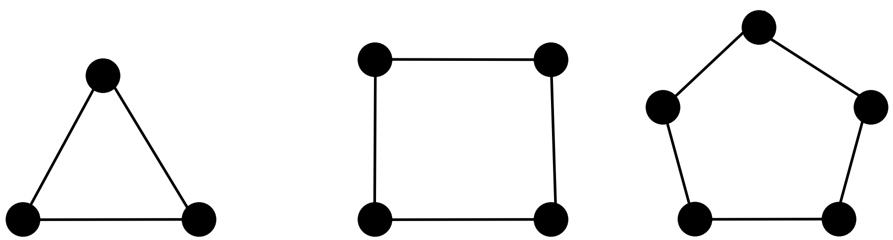


Note. K_n is $(n-1)$-regular, $|V(K_n)|=n$,

$$|E(K_n)| = \binom{n}{2}$$

Some Special Simple Graphs

- The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1,v_2\}, \{v_2,v_3\}, \ldots, \{v_{n-1},v_n\}, \{v_n,v_1\}$.



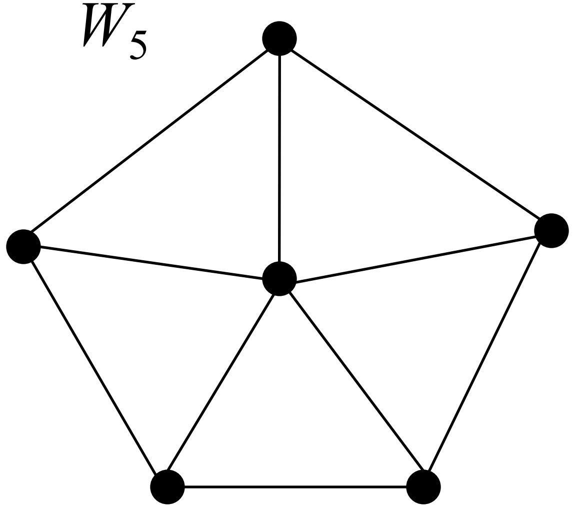
C_3 C_4 C_5

Note C_n is 2-regular, $|V(C_n)| = n$, $|E(C_n)| = n$

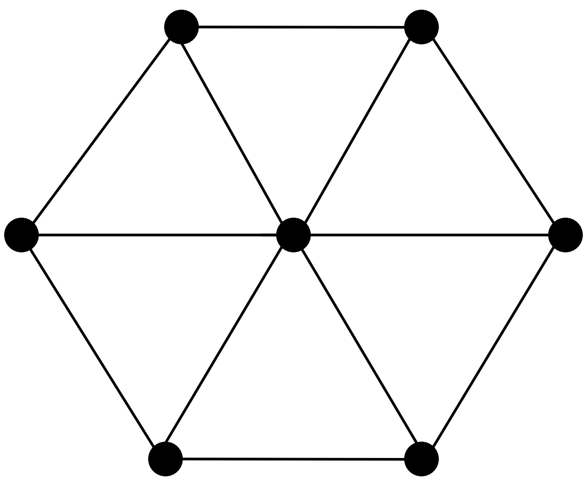
Some Special Simple Graphs

- We obtained the **wheel** W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

W_5



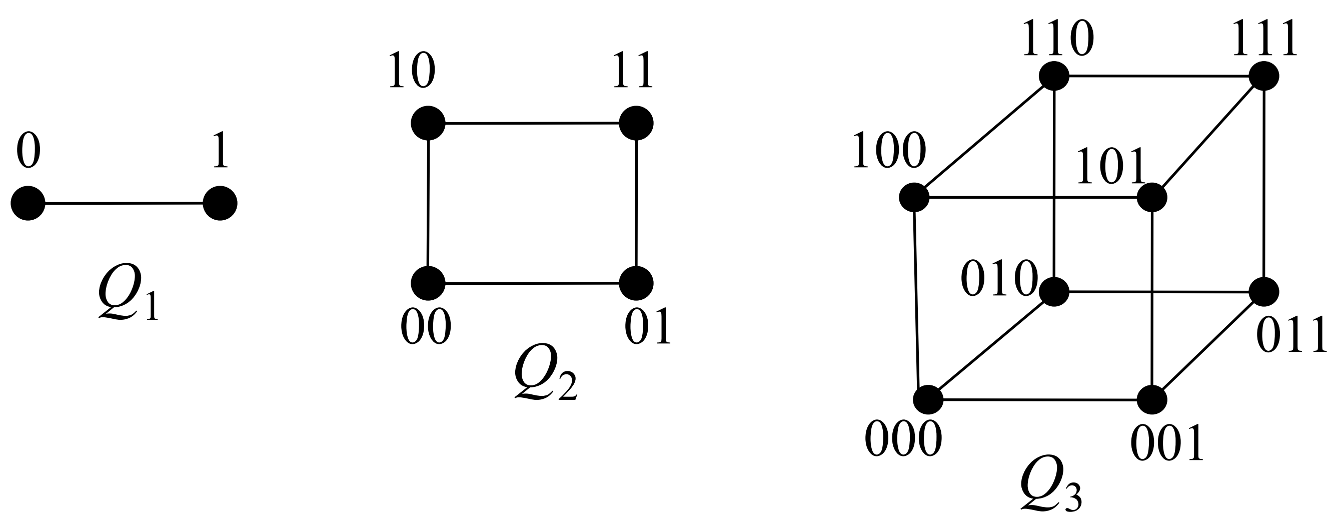
W_6



Note. $|V(W_n)| = n + 1$, $|E(W_n)| = 2n$,
W_n is not a regular graph if $n \neq 3$.

Some Special Simple Graphs

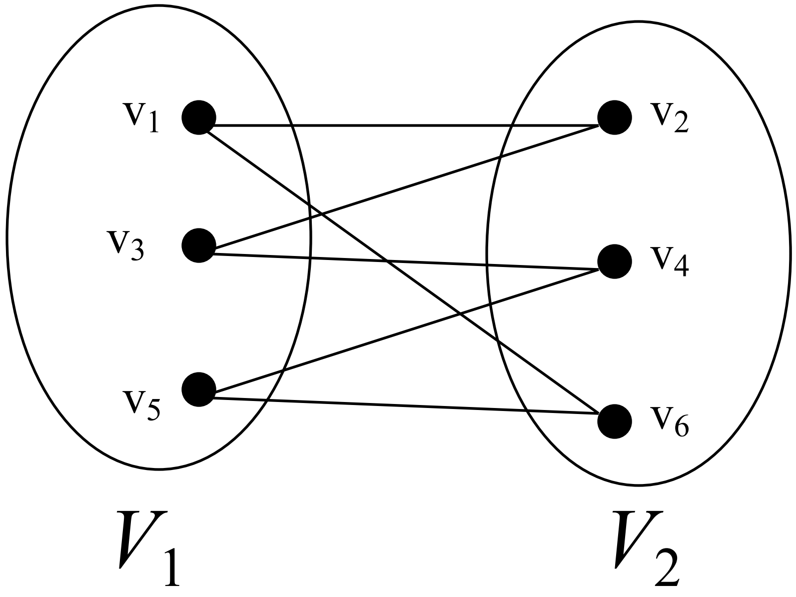
- The n-dimensional hypercube, or n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n.
  Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.



Note. Q_n is n-regular, $|V(Q_n)| = 2^n$, $|E(Q_n)| = (2^n n)/2 = 2^{n-1} n$

Some Special Simple Graphs: Bipartite

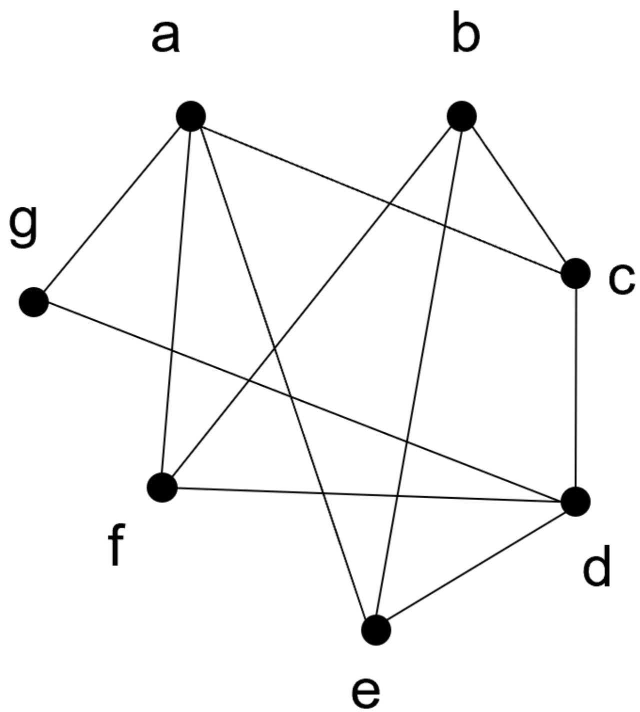
- A simple graph $G=(V,E)$ is called bipartite if V can be partitioned into V_1 and V_2, $V_1 \cap V_2 = \emptyset$, such that every edge in the graph connect a vertex in V_1 and a vertex in V_2.



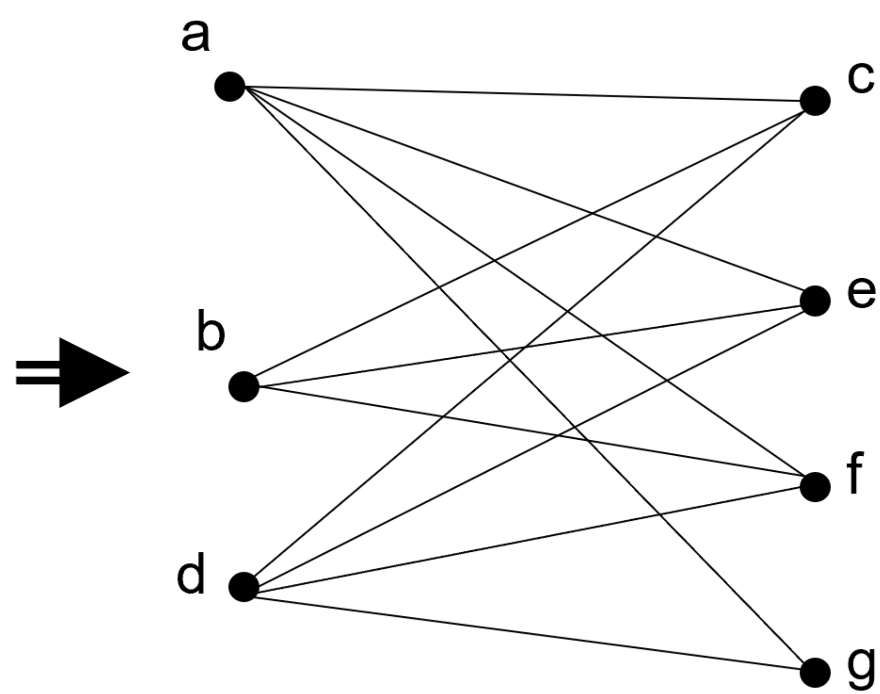
∴ C_6 is bipartite.

Some Special Simple Graphs: Bipartite

- Is the graph G bipartite ?



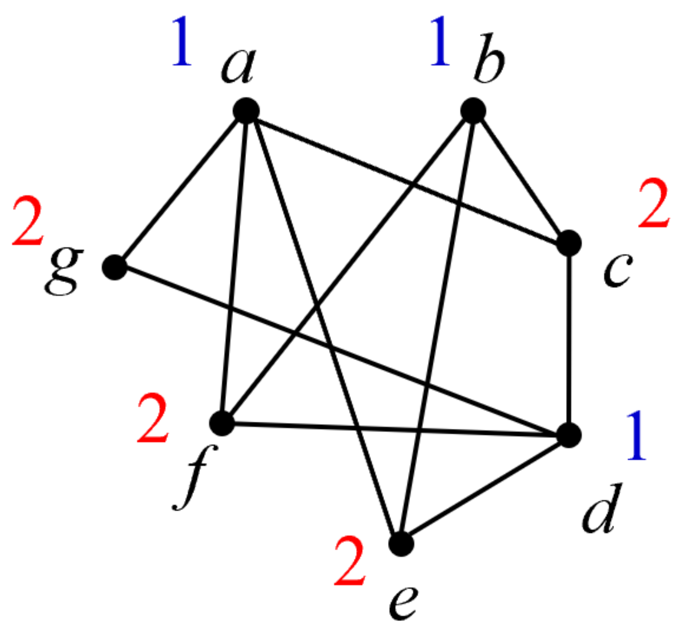
G



Yes !

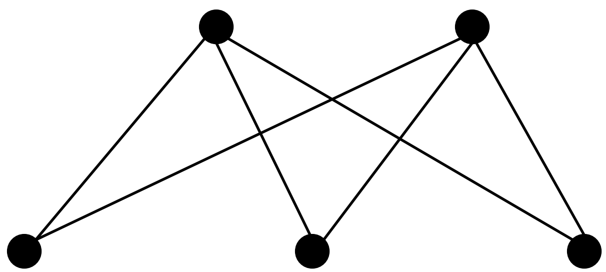
Some Special Simple Graphs: Bipartite

- A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

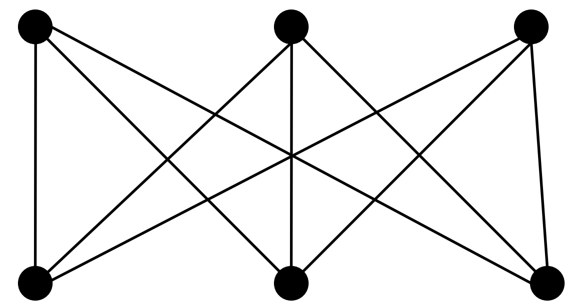


Some Special Simple Graphs: Bipartite

- Complete Bipartite graphs ($K_{m,n}$)



$K_{2,3}$

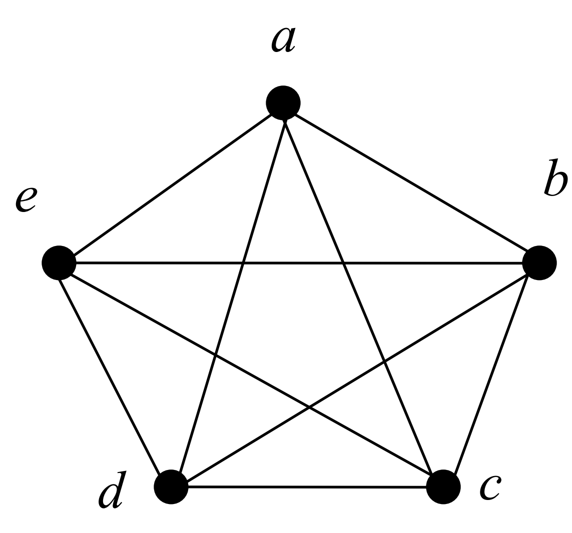


$K_{3,3}$

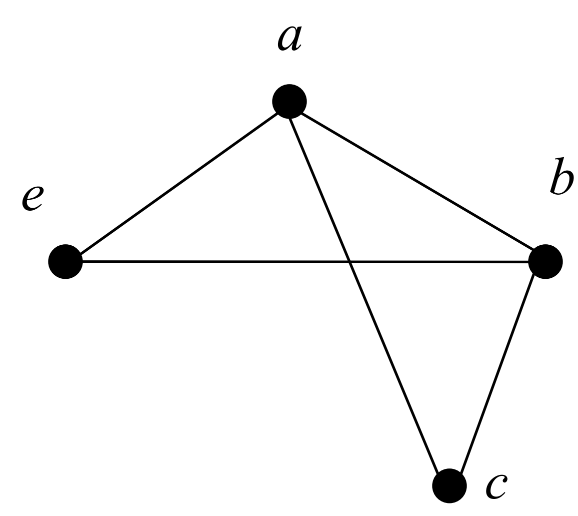
Note. $|V(K_{m,n})| = m+n$, $|E(K_{m,n})| = mn$,
$K_{m,n}$ is regular if and only if $m=n$.

New Graphs from Old

- A subgraph of a graph $G=(V, E)$ is a graph $H=(W, F)$ where $W \subseteq V$ and $F \subseteq E$. (Notice the f point w to connect)



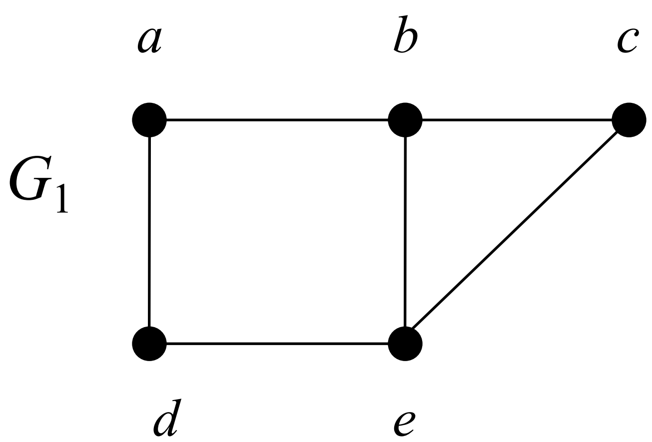
K_5

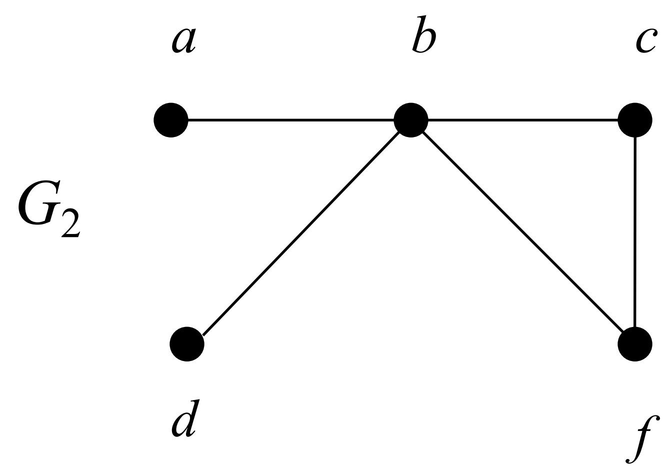


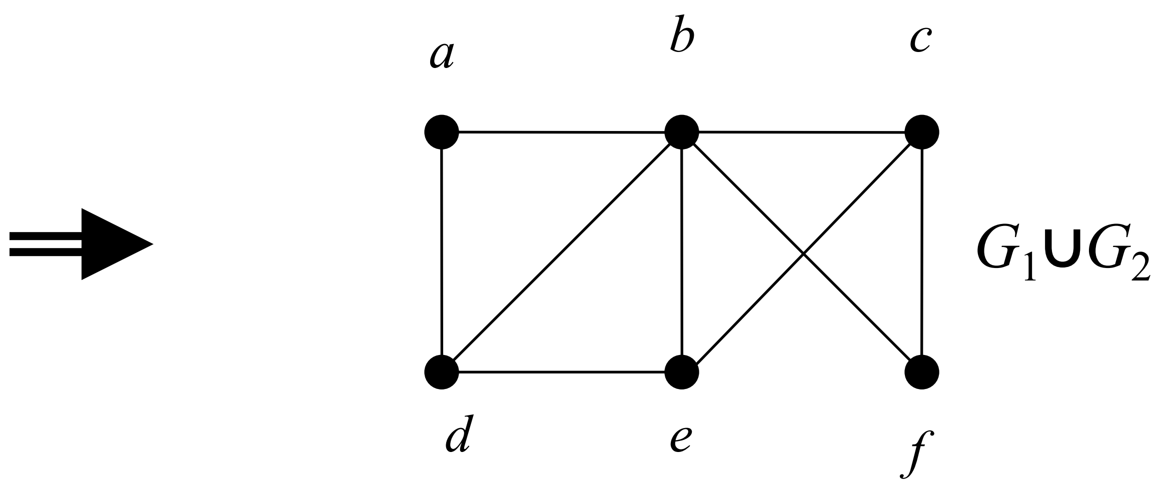
subgraph of K_5

The union of two simple graphs

- $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is the simple graph
  $G_1 \cup G_2=(V_1 \cup V_2, E_1 \cup E_2)$







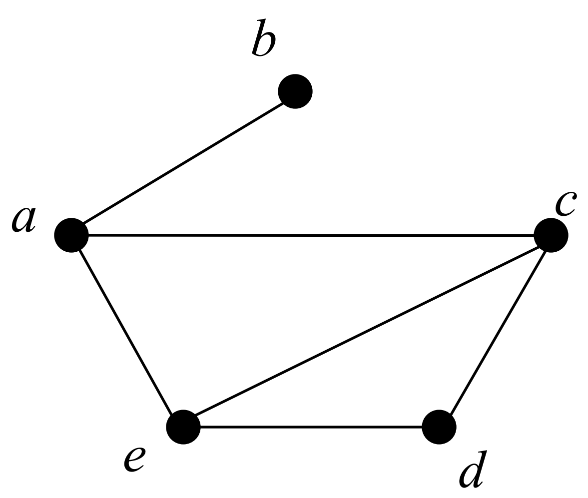
Outline

- Graph and Graph Models
- Graph Terminology and Special Types of Graphs
- **Representing Graphs and Graph Isomorphism**

Representing Graphs and Graph Isomorphism

Adjacency list: Undirected graph

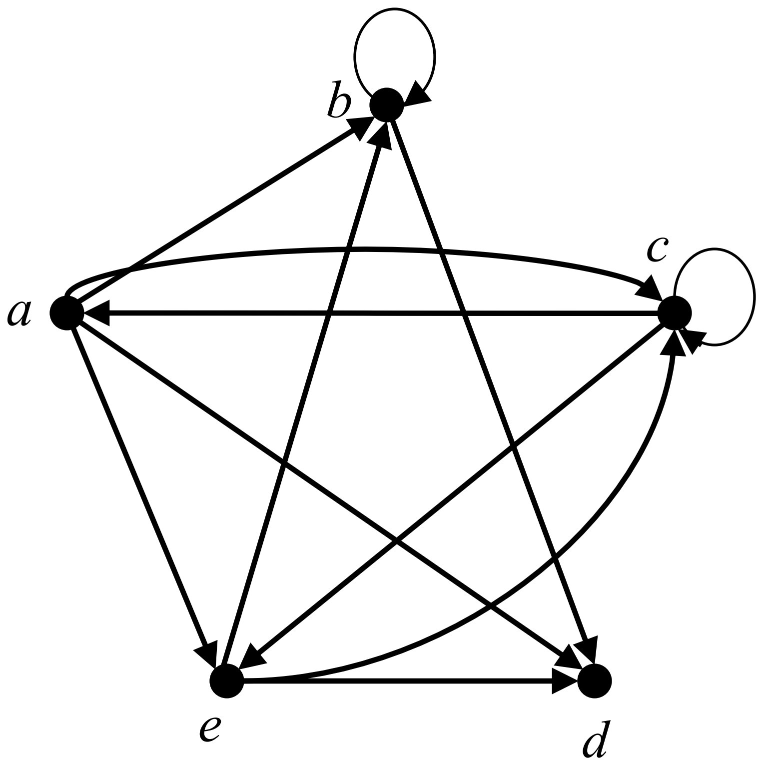
Example: Use adjacency lists to describe the simple graph given below.



Vertex	Adjacent Vertices
a	*b,c,e*
b	*a*
c	*a,d,e*
d	*c,e*
e	*a,c,d*

Representing Graphs and Graph Isomorphism

- **Adjacency list: Directed graph**



Initial vertex	Terminal vertices
a	b,c,d,e
b	b,d
c	a,c,e
d	
e	b,c,d

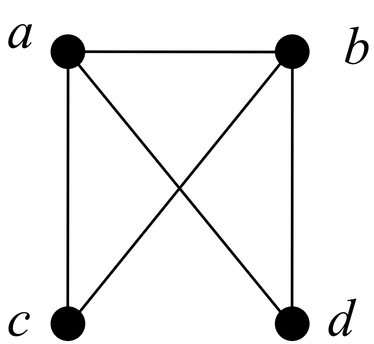
Representing Graphs and Graph Isomorphism

Adjacency Matrices

$G=(V, E)$: simple graph, $V=\{v_1,v_2,\ldots,v_n\}$.

A matrix A is called the adjacency matrix of G

if $A=[a_{ij}]_{n\times n}$, where $a_{ij}=$ 1, if $\{v_i,v_j\}\in E$, and 0 otherwise.



$$A_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ b \\ d \\ c \\ a \end{array} \begin{array}{c} \begin{array}{cccc} b & d & c & a \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

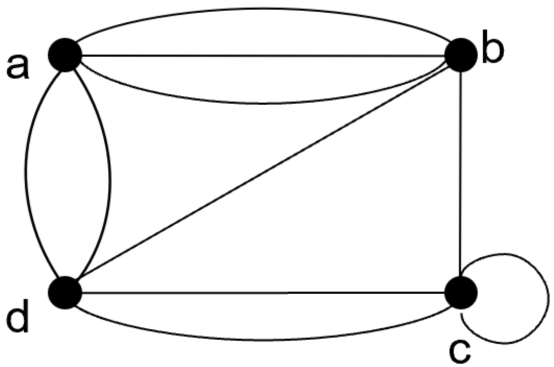
Note:

1. There are $n!$ different adjacency matrices for a graph with n vertices.
2. The adjacency matrix of an undirected graph is symmetric.
3. $a_{ii} = 0$ (simple matrix has no loop)

Representing Graphs and Graph Isomorphism

Adjacency Matrices

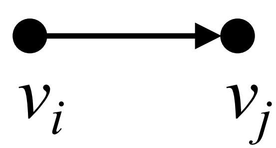
(Pseudograph) (Matrix may not be 0,1 matrix.)



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

If $A=[a_{ij}]$ is the adjacency matrix for the directed graph, then

$$a_{ij} = \begin{cases} 1 & \text{, if } v_i \longrightarrow v_j \\ 0 & \text{, otherwise} \end{cases}$$

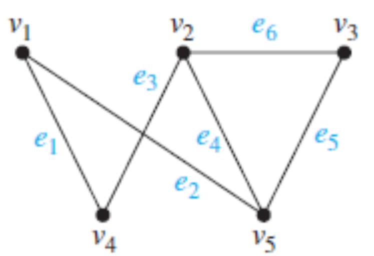


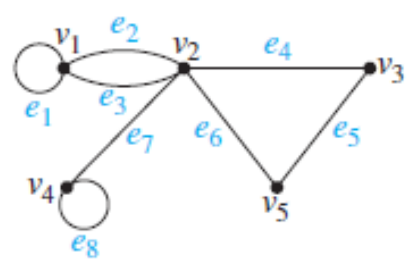
(So the matrix is not necessarily symmetrical)

Representing Graphs and Graph Isomorphism

- **Incidence Matrices**
- Let $G=(V, E)$: be an undirected graph. Suppose that $v_1,v_2,\ldots,v_n$ are the vertices and $e_1,e_2,\ldots,e_n$ are the edges of G . Then the incidence matrix with respect to this ordering of V and E is the n x m matrix $M=[m_{ij}]$, where

$$m_{i,j} = \begin{cases} 1 \text{ when edge } e_j \text{ is incident with } v_i, \\ 0 \text{ otherwise.} \end{cases}$$





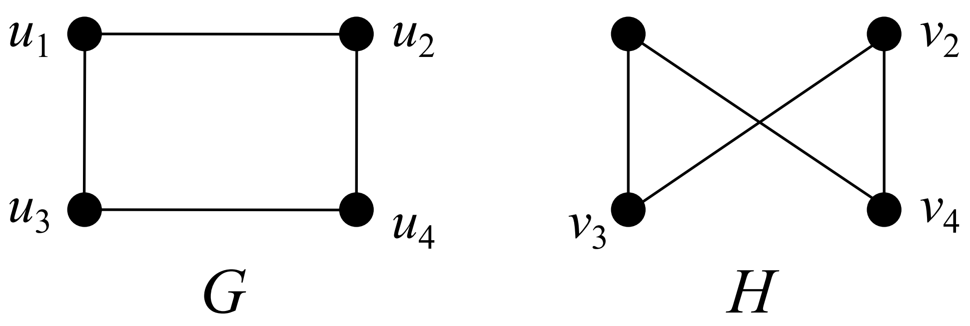
	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	1	0	0	0	0	0
v_2	0	1	1	1	0	1	1	0
v_3	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1
v_5	0	0	0	0	1	1	0	0

Representing Graphs and Graph Isomorphism

- **Isomorphism of Graphs**

The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is an one-to-one and onto function f from V_1 to V_2 with the property that $a \sim b$ in G_1 iff $f(a) \sim f(b)$ in G_2, $\forall a,b \in V_1 f$, f is called an isomorphism.

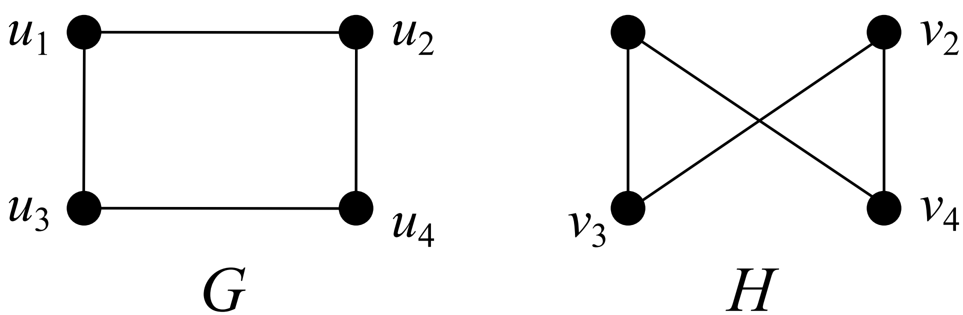


G is isomorphic to H

Representing Graphs and Graph Isomorphism

- Show that G and H are isomorphic.

Solution: The function f with $f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3,$ and $f(u_4) = v_2$ is a one-to-one correspondence between $V(G)$ and $V(H)$.



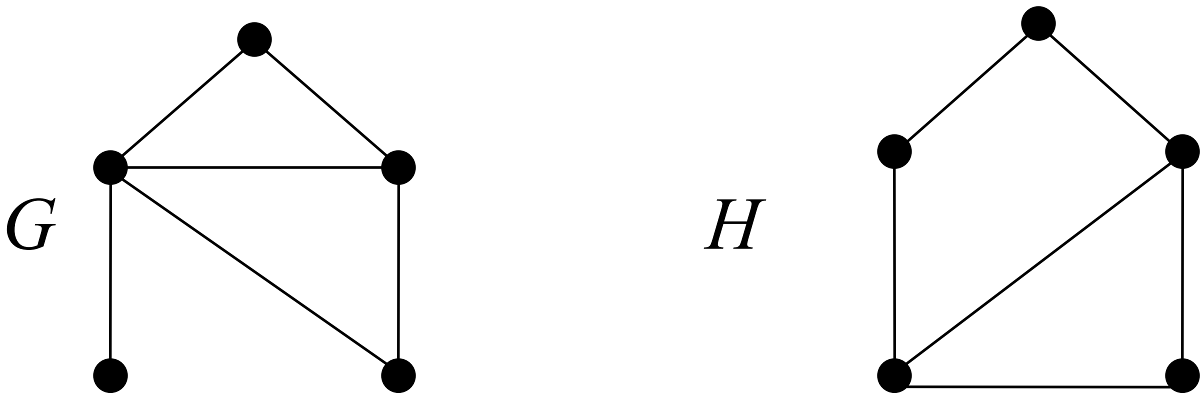
G is isomorphic to H

Isomorphism graphs there will be:

1. The same number of points (vertices)
2. The same number of edges
3. The same number of degree

Representing Graphs and Graph Isomorphism

- Show that G and H are not isomorphic.

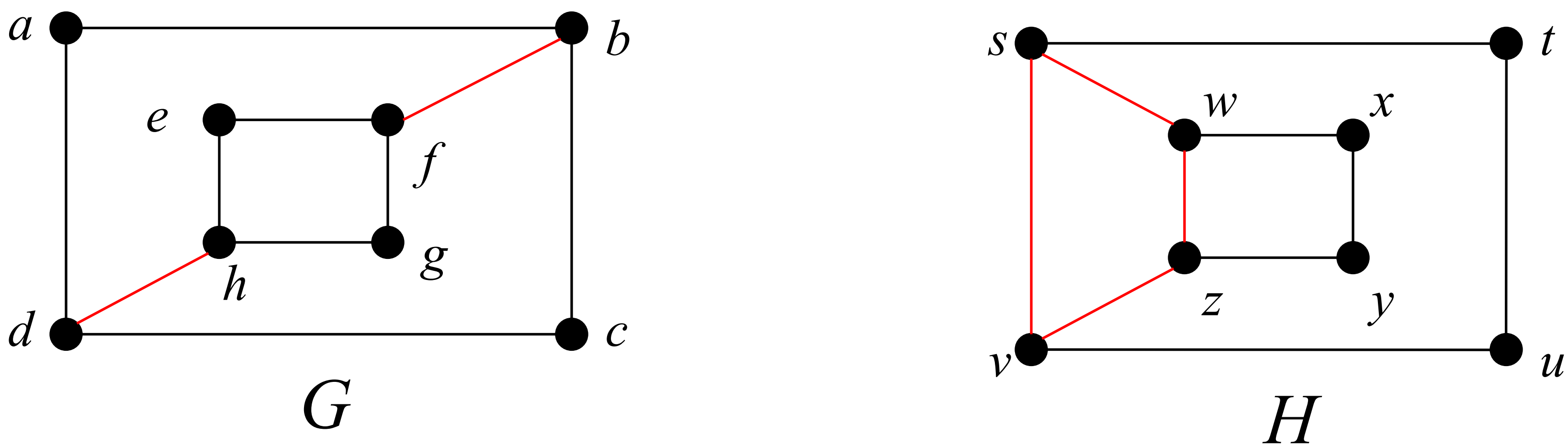


Solution :

G has a vertex of degree = 1 , H don't

Representing Graphs and Graph Isomorphism

- Determine whether G and H are isomorphic.



Solution : ∵ In G, deg(a)=2, which must correspond to either t, u, x, or y in H degree

Each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G

∴ G and H are not isomorphic.

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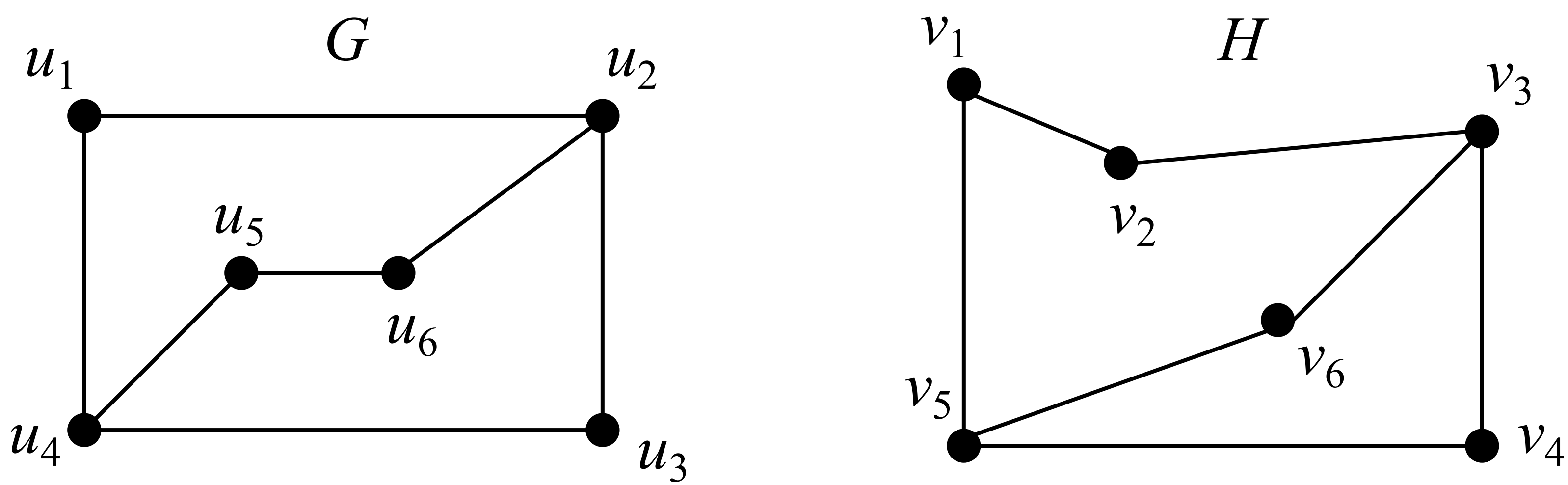

Solution:

$f(u_1)=v_6, f(u_2)=v_3, f(u_3)=v_4, \; f(u_4)=v_5, f(u_5)=v_1, f(u_6)=v_2$

$\Rightarrow$ Yes

Next class

- Topic: Graph Theory (2)
- Pre-class reading: Chap 10.3-10.5