UNIVERSITY AT ALBANY
State University of New York

Lecture 34: Graph Theory (2)

Dr. Chengjiang Long
Computer Vision Researcher at Kitware Inc.
Adjunct Professor at SUNY at Albany.
Email: **clong2@albany.edu**

Outline

- Connectivity
- Euler and Hamiltonian Paths
- Shortest-Path Problems

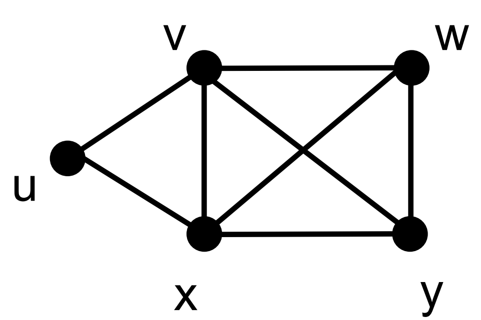
Outline

- **Connectivity**
- Euler and Hamiltonian Paths
- Shortest-Path Problems

Connectivity

- In an undirected graph, a path of length n from u to v is a sequence of $n+1$ adjacent vertices going from vertex u to vertex v.
  (e.g., P: $u=x_0, x_1, x_2, \ldots, x_n=v$.) (P has n edges.)
- path: Points and edges in unrepeatable
  trail: Allows duplicate path (not repeatable)
  walk: Allows point and duplicate path

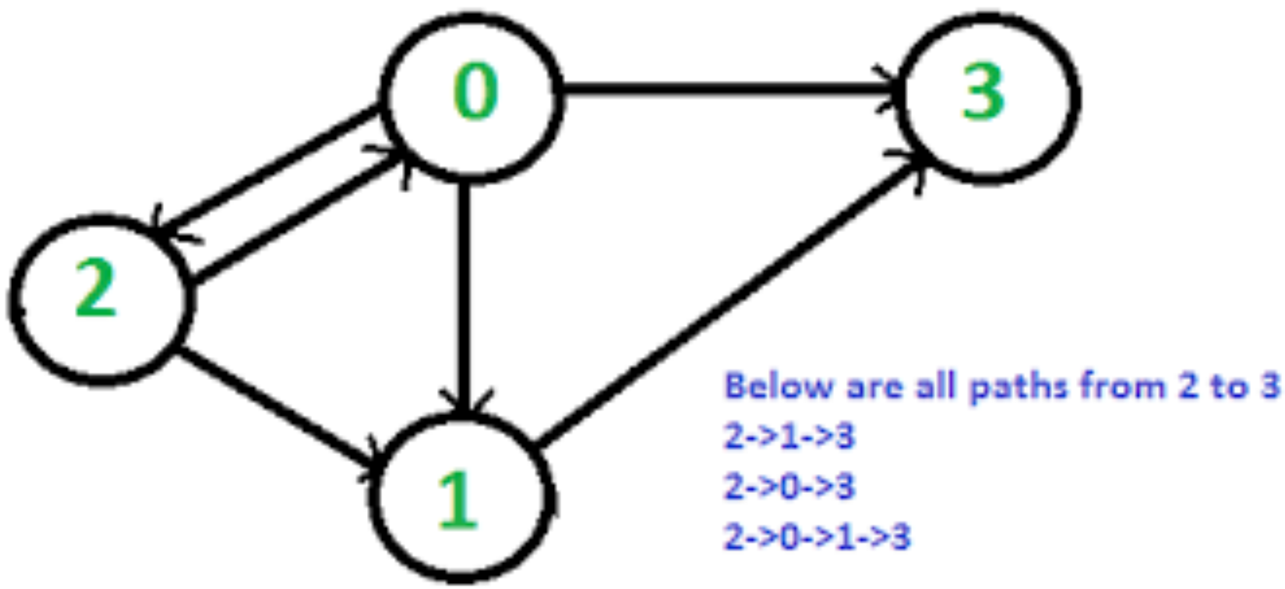
  cycle: path with $u=v$



path: u, v, y
trail: u, v, w, y, v, x, y
walk: u, v, w, v, x, v, y
cycle: u, v, y, x, u

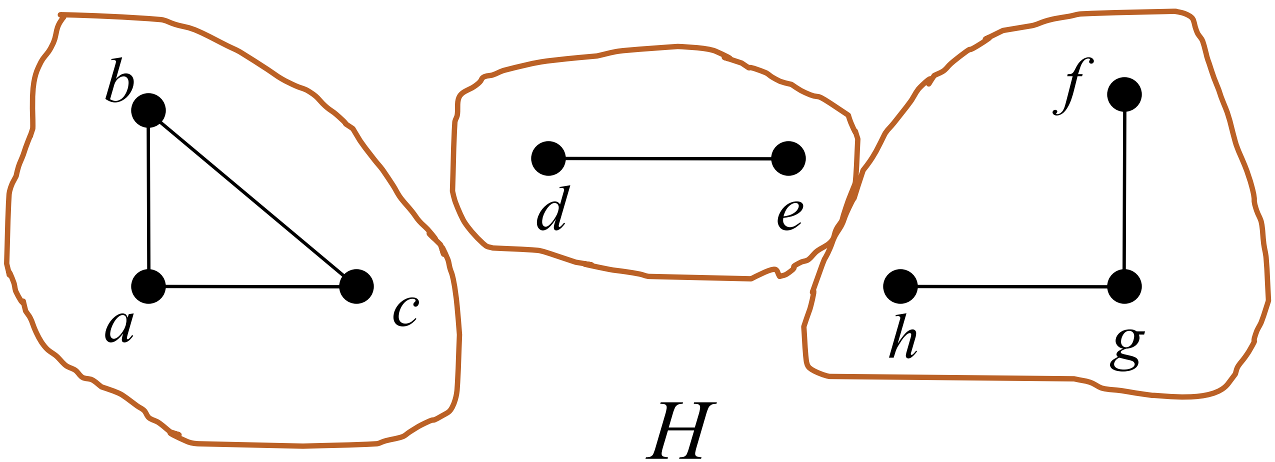
Paths in Directed Graphs

- The same as in undirected graphs, but the path must go in the direction of the arrows.



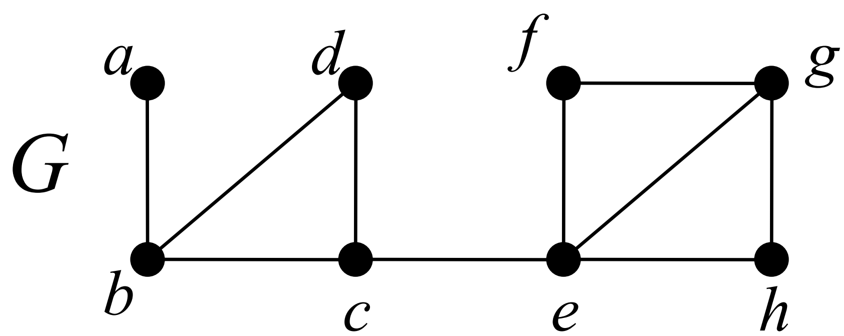
Connectedness in Undirected Graphs

- An undirected graph is *connected* if there is a path between every pair of distinct vertices in the graph.
- *Connected component*: maximal connected subgraph. (An unconnected graph will have several component)



Cut Vertex and Cut Edge

- A *cut vertex* separates one connected component into several components if it is removed.
  A *cut edge* separates one connected component into two components if it is removed.



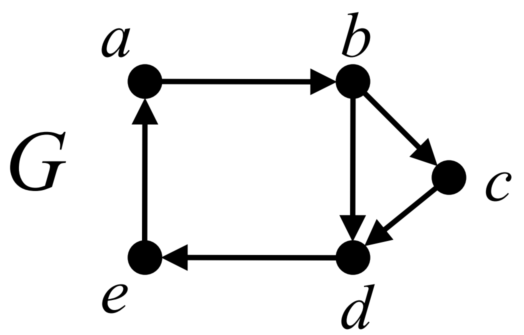
cut vertices: b, c, e

cut edges:

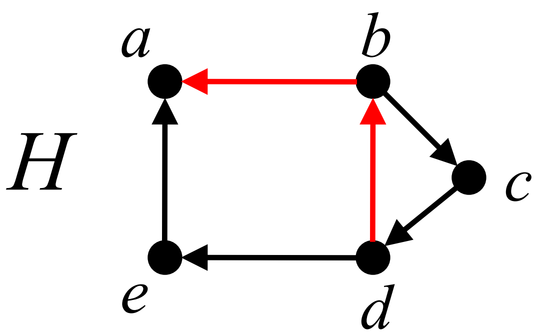
$\{a, b\}$, $\{c, e\}$

Connectedness in Directed Graphs

- A directed graph is *strongly connected* if there is a path from a to b for any two vertices a, b.
  A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graphs.



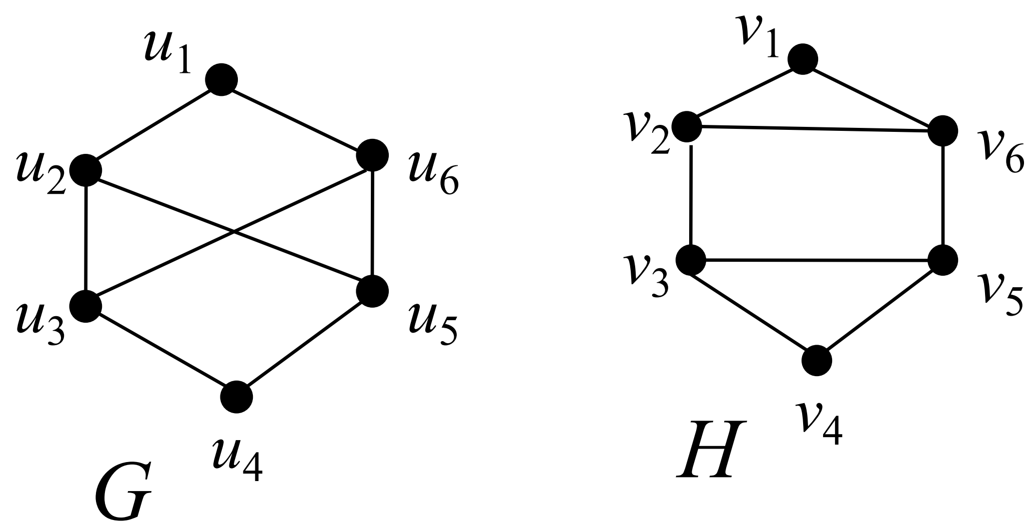
strongly connected



weakly connected

Paths and Isomorphism

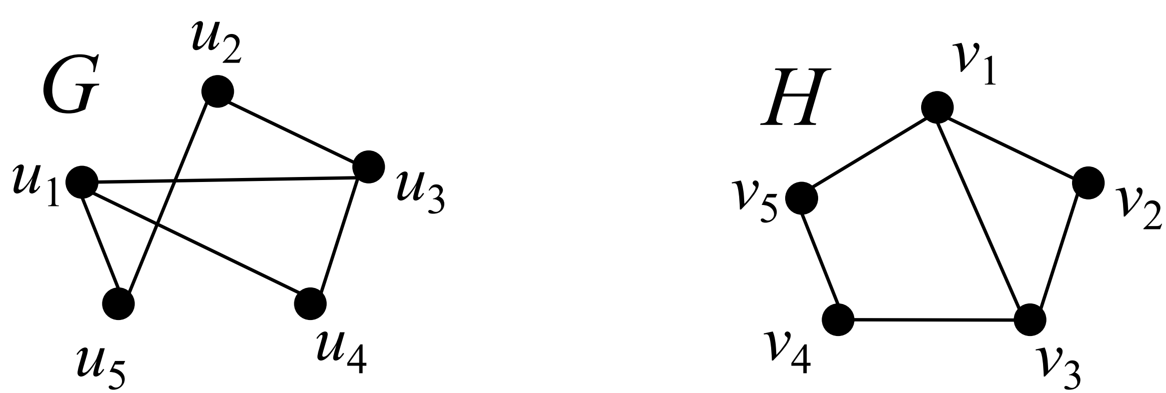
- Note that connectedness, and the existence of a circuit or simple circuit of length *k* are graph invariants with respect to isomorphism.
- Determine whether the graphs G and H are isomorphic.



Solution: No, Because G has no simple circuit of length three, but H does.

Paths and Isomorphism

- Determine whether the graphs G and H are isomorphic.



Solution:

Both G and H have 5 vertices, 6 edges, two vertices of deg 3, three vertices of deg 2, a 3-cycle, a 4-cycle, and a 5-cycle.
$\Rightarrow$ G and H may be isomorphic.
The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$ and $f(u_5) = v_5$ is a one-to-one correspondence between $V(G)$ and $V(H)$. $\Rightarrow$ G and H are isomorphic.

Outline

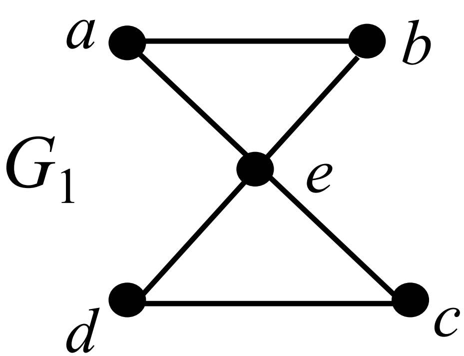
- Connectivity
- **Euler and Hamiltonian Paths**
- Shortest-Path Problems

Euler and Hamilton Paths

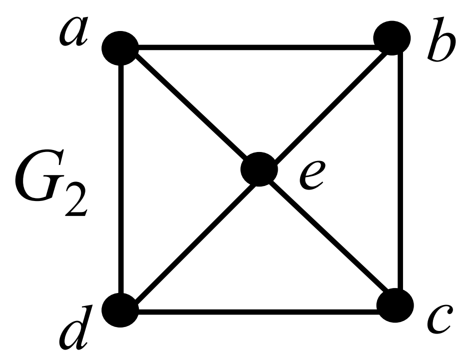
- An *Euler circuit* in a graph *G* is a simple circuit containing every edge of *G*.
  An *Euler path* in *G* is a simple path containing every edge of *G*.
- A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.
- A connected multigraph has an Euler path (but not an Euler circuit) if and only if it has exactly 2 vertices of odd degree.

Example

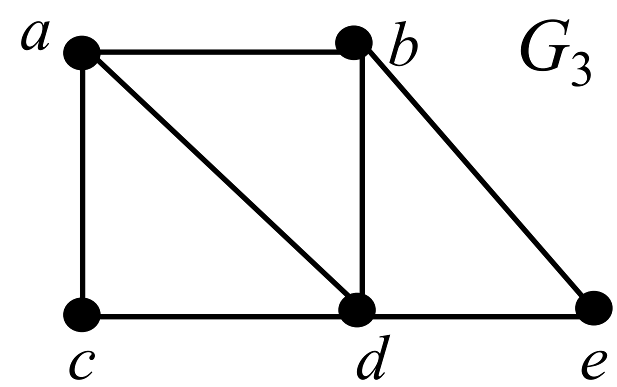
- Which of the following graphs have an Euler circuit or an Euler path?



Euler circuit



none

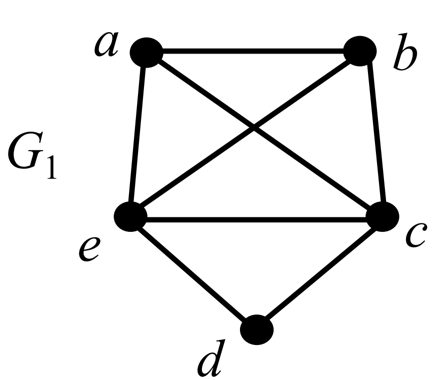


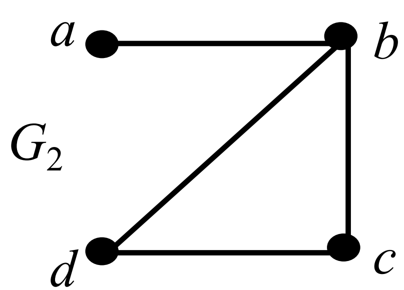
Euler path

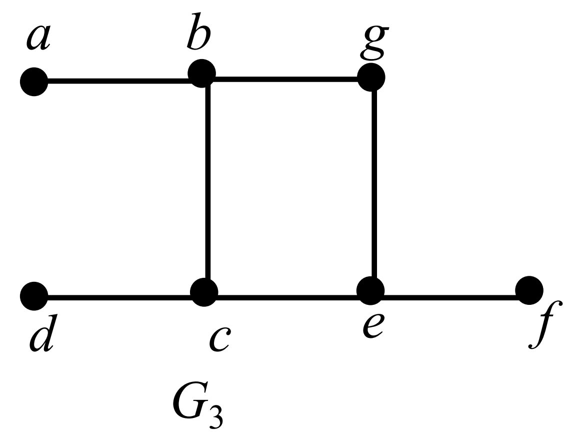
Hamilton Paths and Circuits

- A *Hamilton path* is a path that traverses each <u>vertex</u> in a graph $\boldsymbol{G}$ exactly once.
  A *Hamilton circuit* is a circuit that traverses each vertex in $\boldsymbol{G}$ exactly once.

- Which of the following graphs have a Hamilton circuit or a Hamilton path?







Hamilton circuit: G_1 Hamilton path: G_1,G_2

Outline

- Connectivity
- Euler and Hamiltonian Paths
- **Shortest-Path Problems**

Shortest-Path Problems

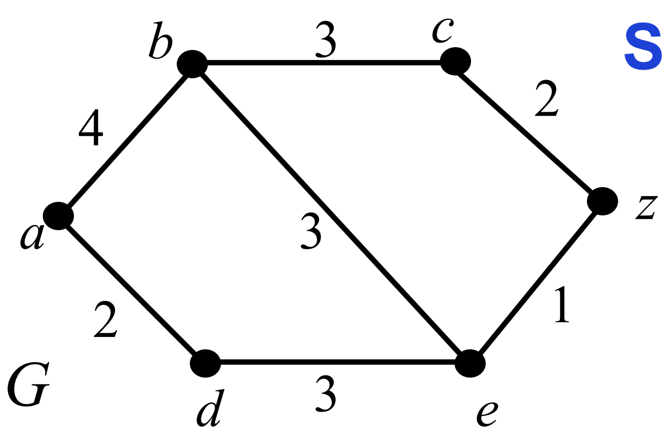
- Graphs that have a number assigned to each edge are called *weighted graphs*.
- The length of a path in a weighted graph is the sum of the weights of the edges of this path.

Shortest path Problem:

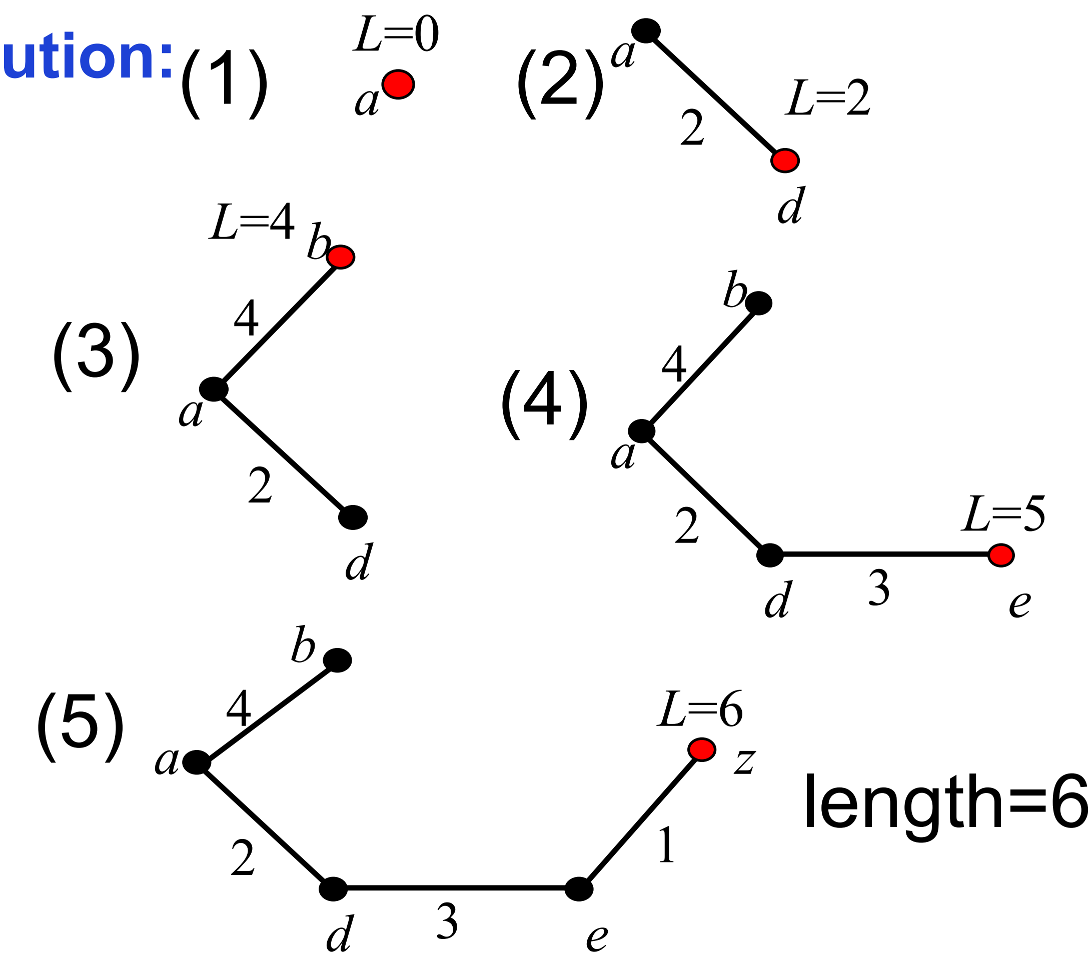
Determining the path of least sum of the weights between two vertices in a weighted graph.

Exampe

- What is the length of a shortest path between a and z in the weighted graph G?



Solution:



Dijkstra's Algorithm

Procedure *Dijkstra*(G: weighted connected simple graph, with all weights positive)

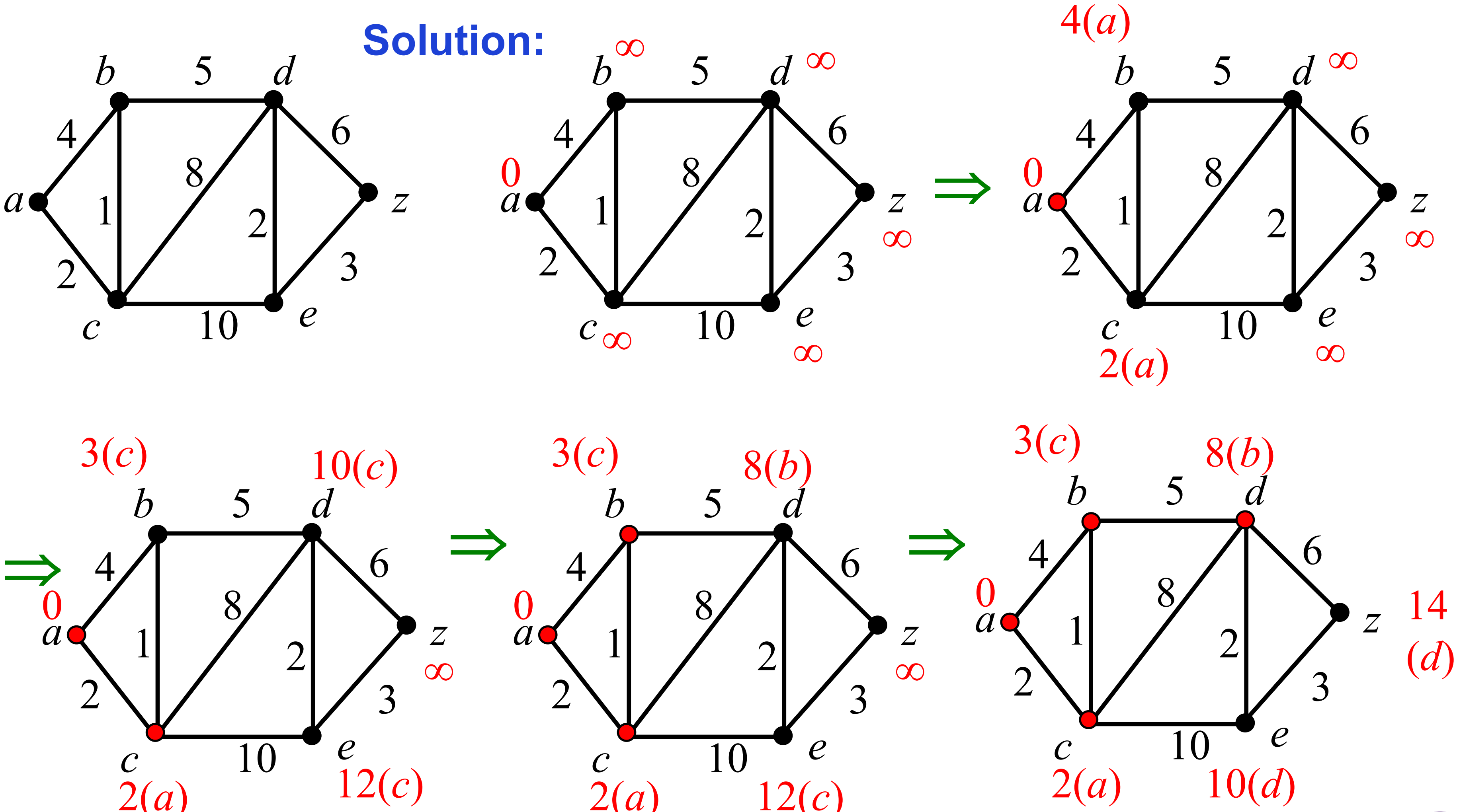
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and weights $w(v_i, v_j)$ where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}

for $i := 1$ **to** n
    $L(v_i) := \infty$
$L(a) := 0$
$S := \varnothing$
while $z \notin S$
begin
    $u :=$ a vertex not in S with $L(u)$ minimal
    $S := S \cup \{u\}$
    for all vertices v not in S
        if $L(\text{u}) + w(u, v) < L(v)$ **then** $L(v) := L(\text{u}) + w(u, v)$
end {$L(z)$ = length of a shortest path from a to z}

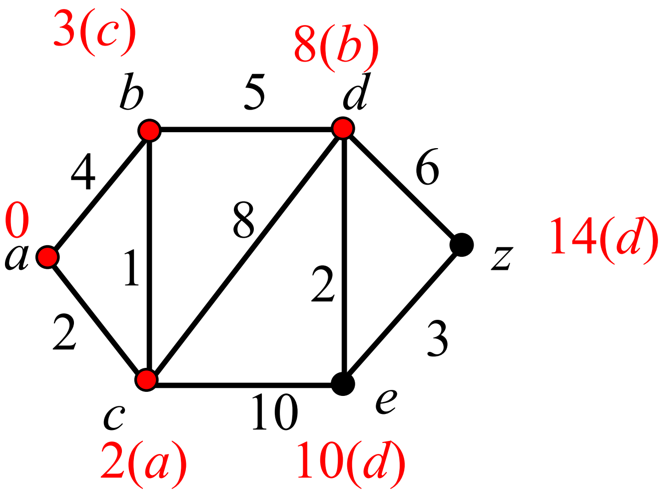
Example

- Use Dijkstra's algorithm a shortest path between a and z.

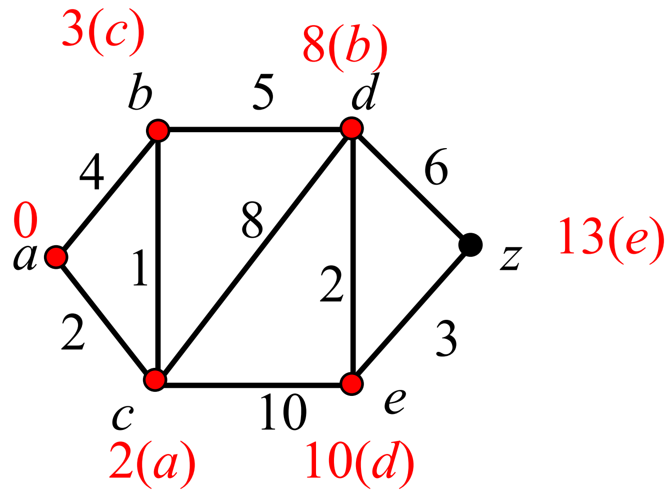
Solution:



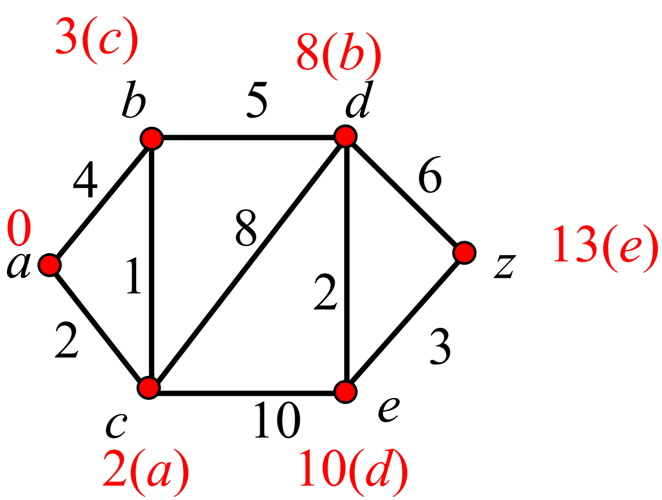
Example: Cont.



⇒



⇒



⇒ path: a, c, b, d, e, z
length: 13

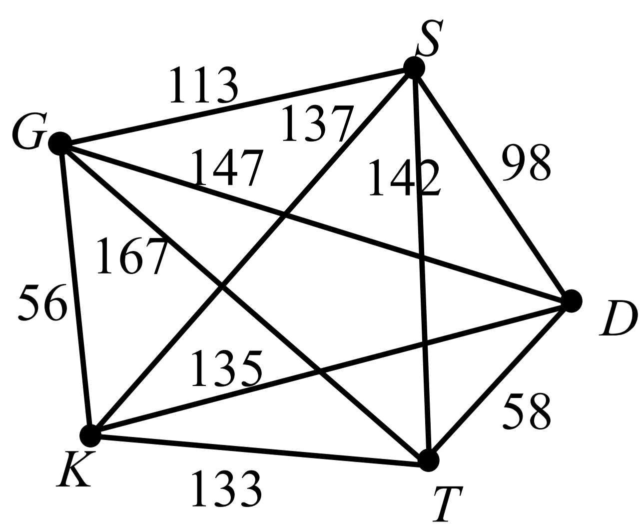
Remarks on Dijkstra's Algorithm

- Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.
- Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with n vertices.

The Traveling Salesman Problem

- A traveling salesman wants to visit each of n cities exactly once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?

Example (starting point D)



$D \to T \to K \to G \to S \to D$: 458

$D \to T \to S \to G \to K \to D$: 504

$D \to T \to S \to K \to G \to D$: 540

…

Next class

- Topic: Graph Theory
- Pre-class reading: Chap 10.1-10.5