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Lecture 6: Introduction to Sets

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Outline

- Definitions: Set, Element
- Terminology and Notation
- Proving Equivalences
- Power Set
- Tuples
- Cartesian Product
- Quantifiers

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- **Definitions: Set, Element**
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Introduction (1)

- We have already implicitly dealt with sets
  - Integers (Z), rationals (Q), naturals (N), reals (R), etc.
- We will develop more fully
  - The definitions of sets
  - The properties of sets
  - The operations on sets
- **Definition**: A set is an unordered collection of (unique) objects
- Sets are fundamental discrete structures and for the basis of more complex discrete structures like graphs

Introduction (2)

- **Definition**: The objects in a set are called <u>elements</u> or <u>members</u> of a set. A set is said to contain its elements
- Notation, for a set A:
  - $x \in A$: x is an element of A — $\in$
  - $x \notin A$: x is not an element of A — $\notin$

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Terminology (1)

- **Definition**: Two sets, A and B, are equal is they contain the same elements. We write A=B.
- Example:
  - {2,3,5,7}={3,2,7,5}, because a set is unordered
  - Also, {2,3,5,7}={2,2,3,5,3,7} because a set contains unique elements
  - However, $\{2,3,5,7\} \neq \{2,3\}$ $\neq$

Terminology (2)

- A multi-set is a set where you specify the number of occurrences of each element: $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ is a set where
  - m_1 occurs a_1 times
  - m_2 occurs a_2 times
  - …
  - m_r occurs a_r times
- In Databases, we distinguish
  - A set: elements cannot be repeated
  - A bag: elements can be repeated

Terminology (3)

- The **set-builder** notation

$$O=\{ x \mid (x \in Z) \wedge (x=2k) \text{ for some } k \in Z\}$$

reads: O is the set that contains all x such that x is an integer and x is even

- A set is defined in **intension** when you give its set-builder notation

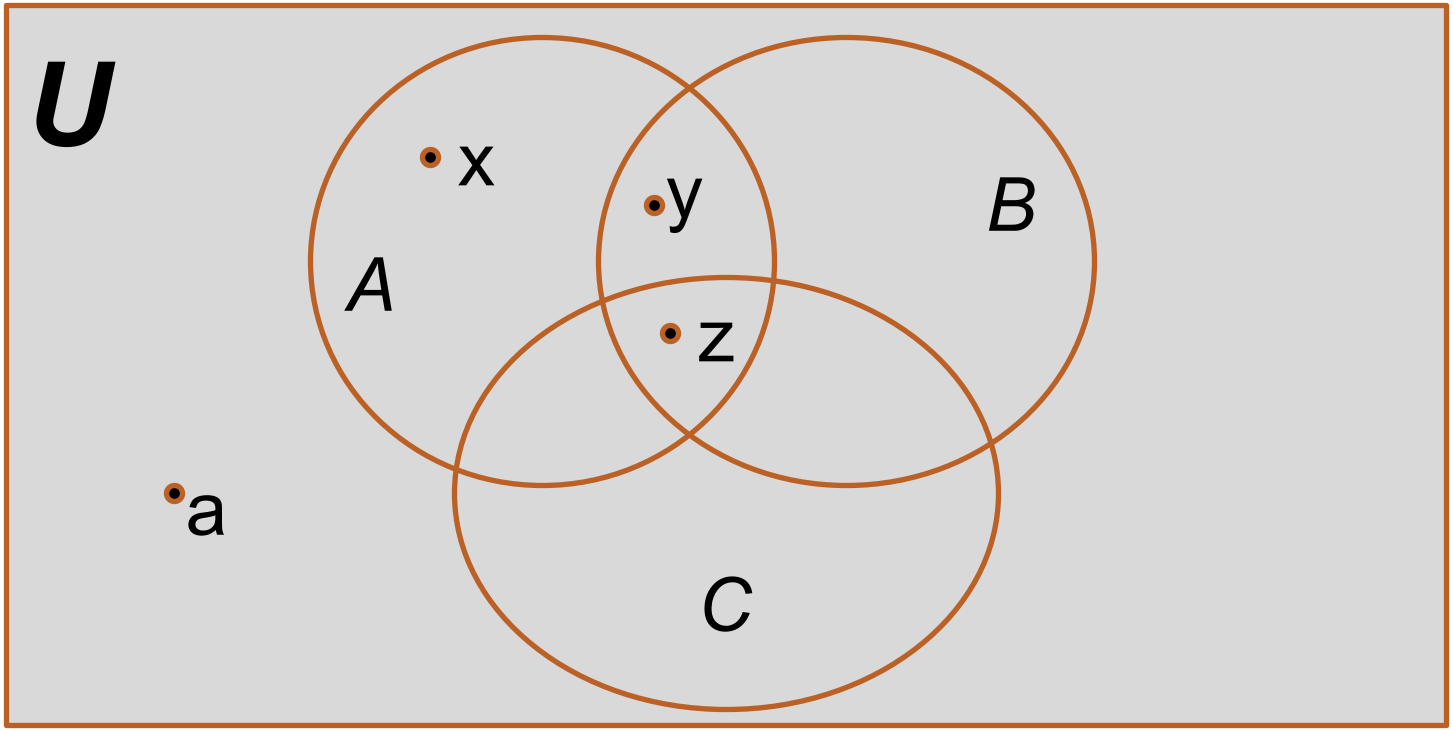
$$O=\{ x \mid (x \in Z) \wedge (0 \leq x \leq 8) \wedge (x=2k) \text{ for some } k \in Z \}$$

- A set is defined in **extension** when you enumerate all the elements:

$$O=\{0,2,4,6,8\}$$

Venn Diagram: Example

- A set can be represented graphically using a Venn Diagram



More Terminology and Notation (1)

- A set that has no elements is called the **empty set** or **null set** and is denoted $\emptyset$ \$\emptyset\$
- A set that has one element is called a **singleton set**.
  - For example: {a}, with brackets, is a singleton set
  - a, without brackets, is an element of the set {a}
- Note the subtlety in $\emptyset \neq \{\emptyset\}$
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set

More Terminology and Notation (2)

- **Definition**: A is said to be a **subset** of B, and we write $A \subseteq B$, if and only if every element of A is also an element of B $\subseteq$ $\$\backslash subseteq\$$
- That is, we have the equivalence:

$$A \subseteq B \Leftrightarrow \forall x\,(x \in A \Rightarrow x \in B)$$

More Terminology and Notation (3)

- **Theorem**: For any set S *Theorem 1, page 115*
  - $\emptyset \subseteq S$ and
  - $S \subseteq S$
- The proof is in the book, an excellent example of a vacuous proof

More Terminology and Notation (4)

- **Definition**: A set A that is a subset of a set B is called a **proper subset** if $A \neq B$.
- That is there is an element $x \in B$ such that $x \notin A$
- We write: $A \subset B$, $A \subsetneq B$
- In LaTex: $\subset$, $\subsetneq$

More Terminology and Notation (5)

- Sets can be elements of other sets
- Examples
  - $S_1 = \{\emptyset,\{a\},\{b\},\{a,b\},c\}$
  - $S_2=\{\{1\},\{2,4,8\},\{3\},\{6\},4,5,6\}$

More Terminology and Notation (6)

- **Definition**: If there are exactly n distinct elements in a set S, with n a nonnegative integer, we say that:
  - S is a finite set, and
  - The cardinality of S is n. Notation: $|S| = n$.
- **Definition**: A set that is not finite is said to be infinite

More Terminology and Notation (7)

- Examples
  - Let B = {x | ($x \leq 100$) $\wedge$ (x is prime)}, the cardinality of B is |B|=25 because there are 25 primes less than or equal to 100.
  - The cardinality of the empty set is $|\emptyset|=0$
  - The sets N, Z, Q, R are all infinite

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Proving Equivalence (1)

- You may be asked to show that a set is
  - a subset of,
  - proper subset of, or
  - equal to another set.
- To prove that A is a subset of B, use the equivalence discussed earlier $A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$
  - To prove that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element x, $x \in A$ implies that x is also in B.
  - Any proof method can be used.
- To prove that A is a proper subset of B, you must prove
  - A is a subset of B **and**
  - $\exists x\ (x \in B) \land (x \notin A)$

Proving Equivalence (2)

- Finally to show that two sets are equal, it is sufficient to show independently (much like a biconditional) that
  - $A \subseteq B$ and
  - $B \subseteq A$
- Logically speaking, you must show the following quantified statements:

$$(\forall x\ (x \in A \Rightarrow x \in B)) \wedge (\forall x\ (x \in B \Rightarrow x \in A))$$

we will see an example later..

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Power Set (1)

- **Definition**: The power set of a set S, denoted P(S), is the set of all subsets of S.
- Examples
  - Let A={a,b,c}, P(A)={∅,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}
  - Let A={{a,b},c}, P(A)={∅,{{a,b}},{c},{{a,b},c}}
- Note: the empty set ∅ and the set itself are always elements of the power set. This fact follows from Theorem 1 (Rosen, page 115).

Power Set (2)

- The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set
- **Fact**: Let S be a set such that |S|=n, then

$$|P(S)| = 2^n$$

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Tuples (1)

- Sometimes we need to consider ordered collections of objects
- **Definition**: The ordered n-tuple $(a_1,a_2,\ldots,a_n)$ is the ordered collection with the element a_i being the i-th element for $i=1,2,\ldots,n$
- Two ordered n-tuples $(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$ are equal iff for every $i=1,2,\ldots,n$ we have $a_i=b_i$ $(a_1,a_2,\ldots,a_n)$
- A 2-tuple (n=2) is called an ordered pair

Cartesian Product (1)

- **Definition**: Let A and B be two sets. The Cartesian product of A and B, denoted AxB, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$

$$A \times B = \{ (a,b) \mid (a \in A) \land (b \in B) \}$$

- The Cartesian product is also known as the cross product
- **Definition**: A subset of a Cartesian product, $R \subseteq A \times B$ is called a relation. We will talk more about relations in the next set of slides
- Note: $A \times B \neq B \times A$ unless $A=\emptyset$ or $B=\emptyset$ or $A=B$. Find a counter example to prove this.

Cartesian Product (2)

- Cartesian Products can be generalized for any n-tuple
- **Definition**: The Cartesian product of n sets, A_1,A_2, …, A_n, denoted $A_1 \times A_2 \times \ldots \times A_n$, is

$$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i=1,2,\ldots,n \}$$

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Notation with Quantifiers

- Whenever we wrote $\exists x P(x)$ or $\forall x P(x)$, we specified the universe of discourse using explicit English language
- Now we can simplify things using set notation!
- Example
  - $\forall\ x \in R\ (x^2 \geq 0)$
  - $\exists\ x \in Z\ (x^2 = 1)$
  - Also mixing quantifiers:

$$\forall a,b,c \in R\ \exists\ x \in C\ (ax^2+bx+c=0)$$

Next class

- Topic: Set Operations and Introduction to Functions
- Pre-class reading: Chap 2.2-2.3