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Lecture 7: Set Operations And Introduction to Functions

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Outline

- Set Operations
- Set Equivalences
- Generalized Unions and Intersection
- Computer Representation of Sets
- Introduction to Functions

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- **Set Operations**
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- Generalized Unions and Intersection
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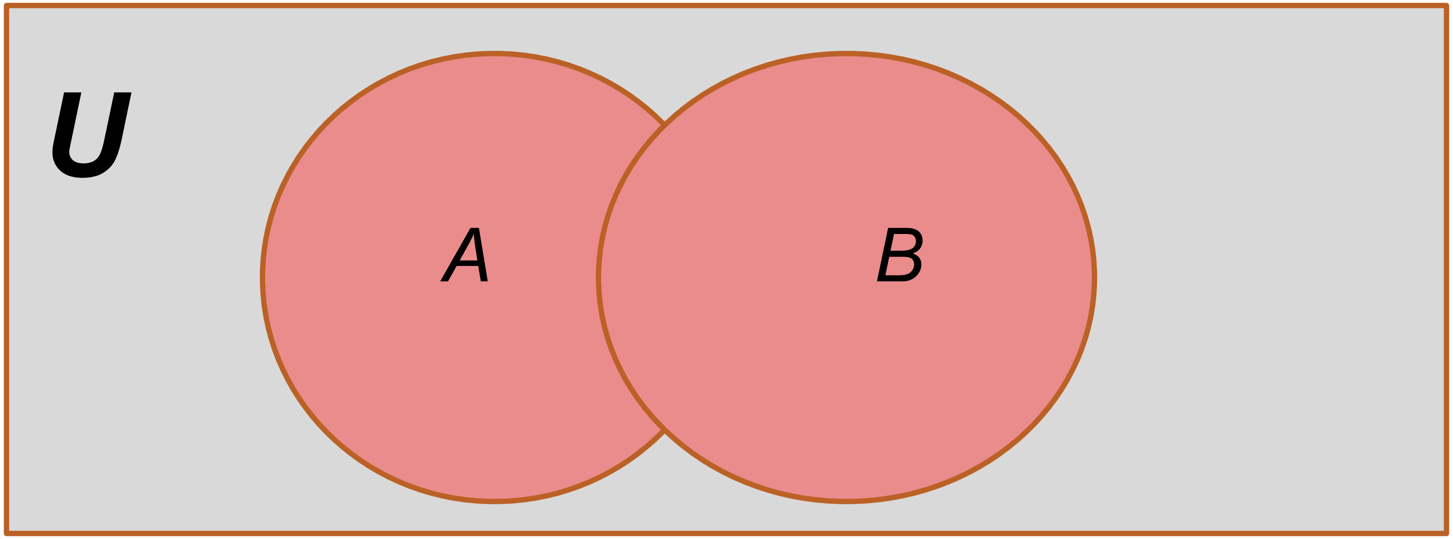
Set Operations

- Arithmetic operators (+,-, × ,÷) can be used on pairs of numbers to give us new numbers
- Similarly, set operators exist and act on two sets to give us new sets
  - Union $\cup$
  - Intersection $\cap$
  - Set difference $\setminus$
  - Set complement $\overline{S}$
  - Generalized union $\bigcup$
  - Generalized intersection $\bigcap$

Set Operators: Union

- **Definition**: The union of two sets A and B is the set that contains all elements in A, B, r both. We write:

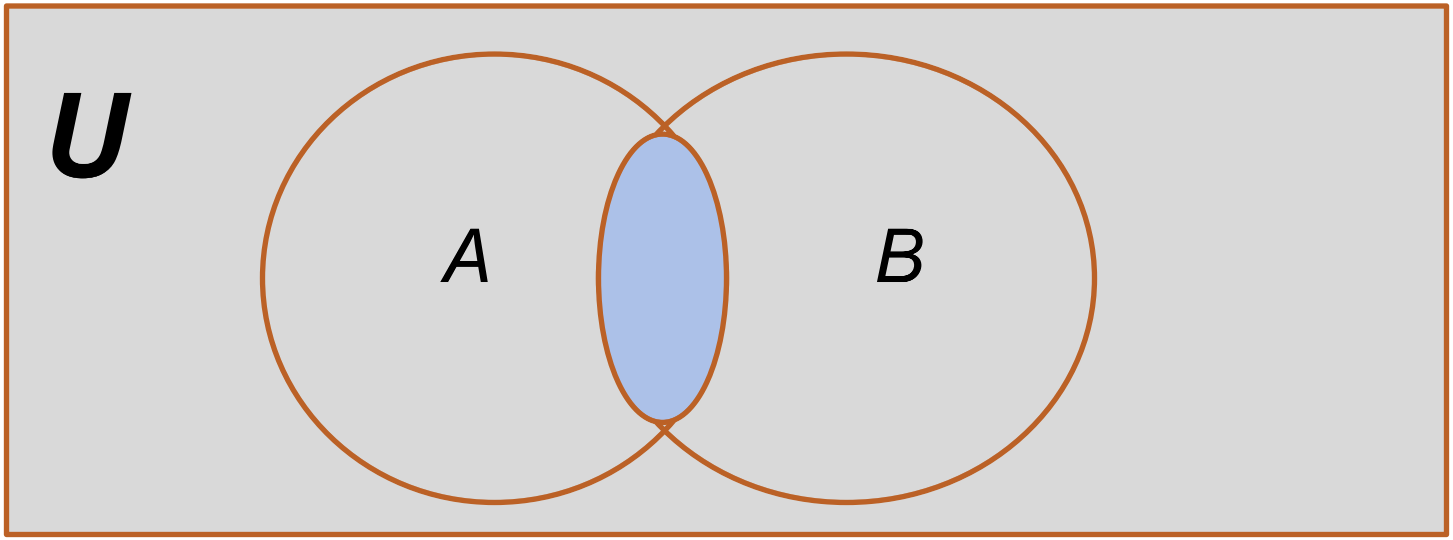
$$A \cup B = \{ x \mid (a \in A) \vee (b \in B) \}$$



Set Operators: Intersection

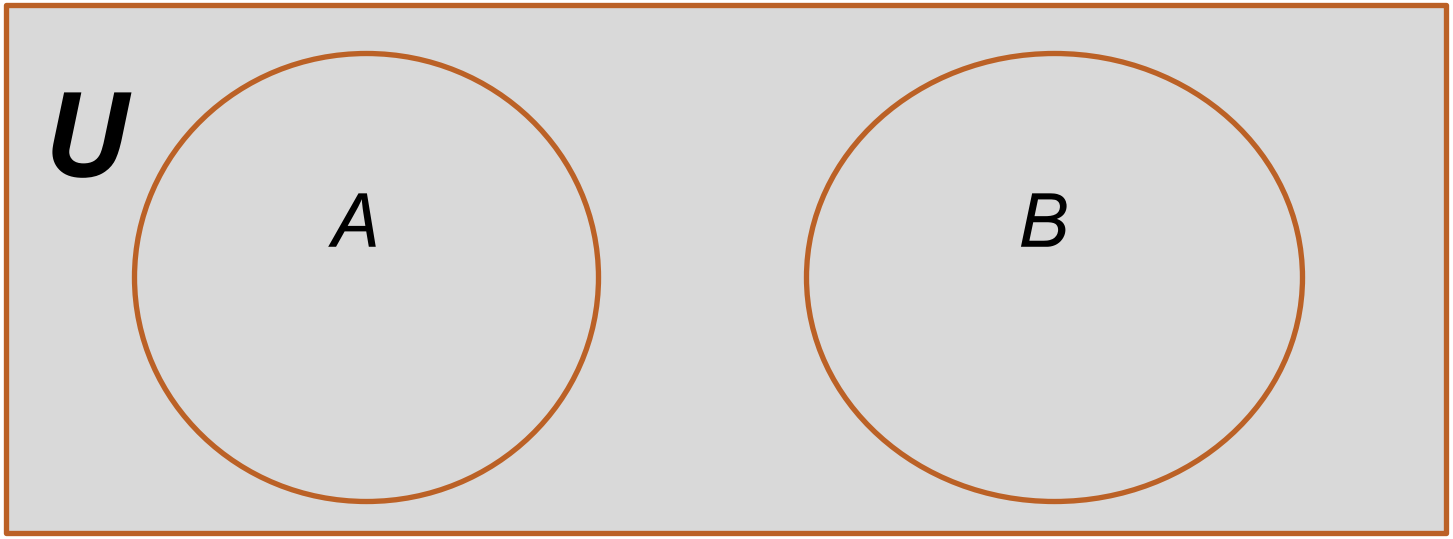
- **Definition**: The intersection of two sets A and B is the set that contains all elements that are element of both A and B. We write:

$$A \cap B = \{ x \mid (a \in A) \wedge (b \in B) \}$$



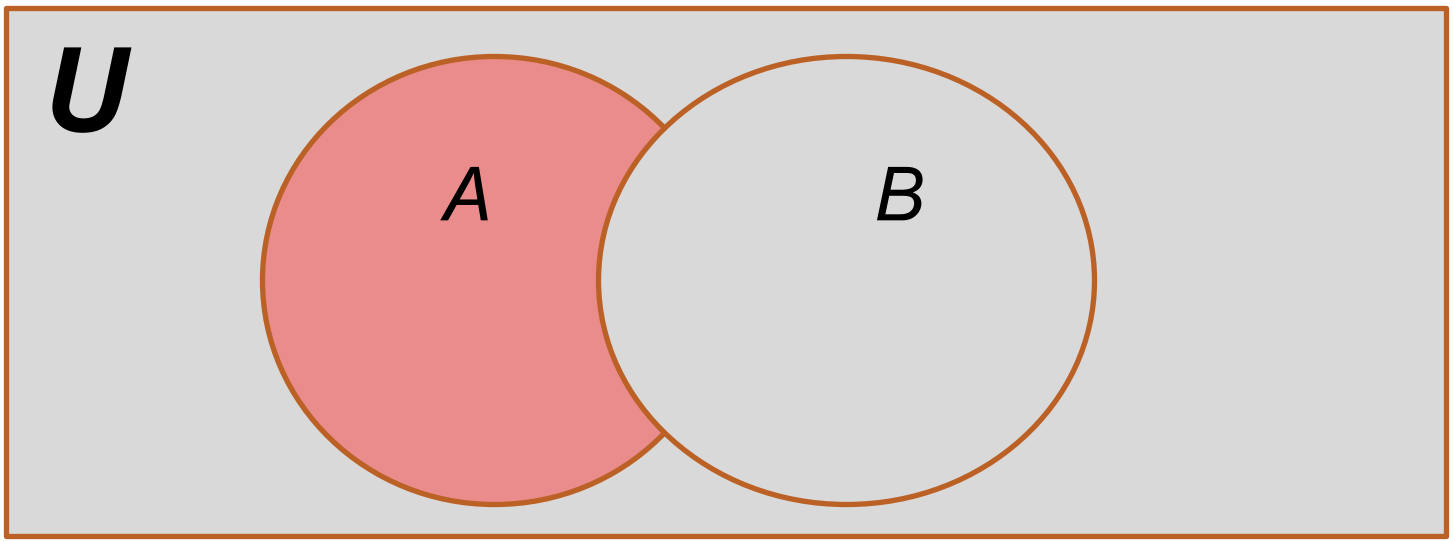
Disjoint Sets

- **Definition**: Two sets are said to be disjoint if their intersection is the empty set: $A \cap B = \varnothing$



Set Difference

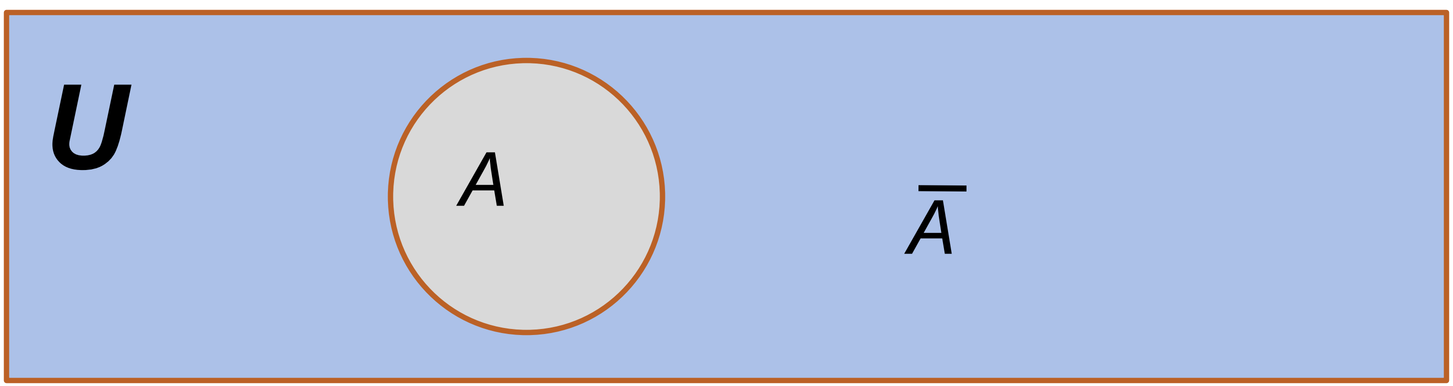
- **Definition**: The difference of two sets A and B, denoted A\B ($\setminus$) or A−B, is the set containing those elements that are in A but not in B



Set Complement

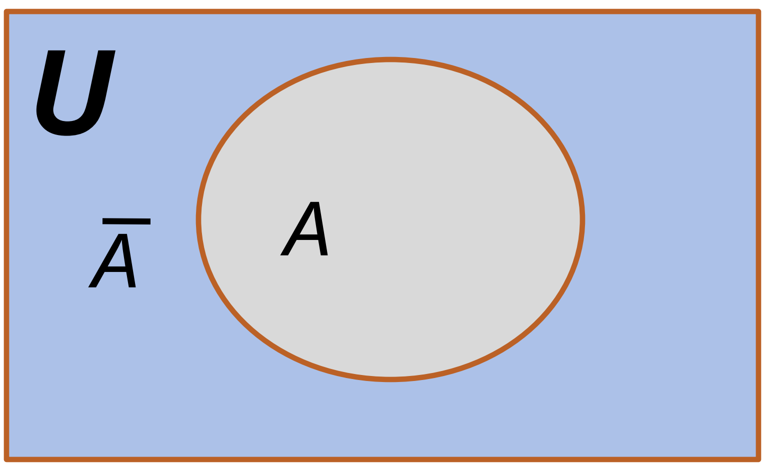
- **Definition**: The complement of a set A, denoted $\bar{A}$ (\$\bar\$), consists of all elements <u>not</u> in A. That is the difference of the universal set and U: U\A

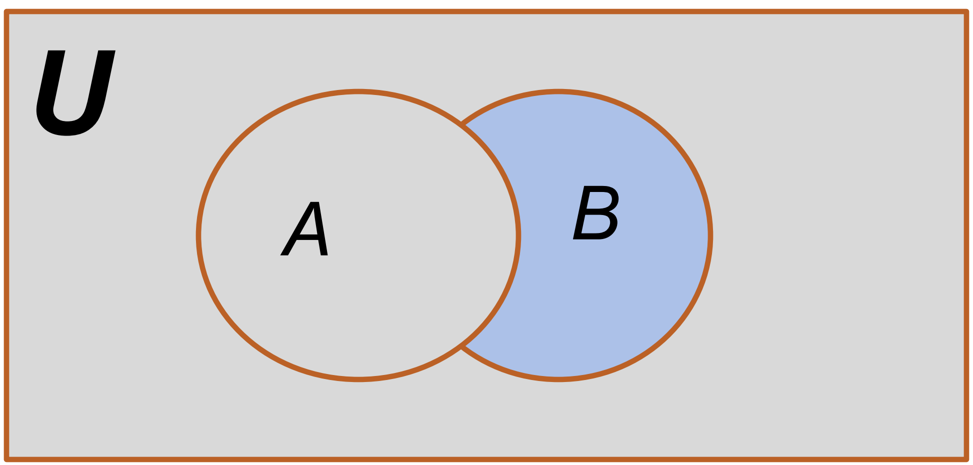
$$\bar{A} = A^C = \{x \mid x \notin A\}$$



Set Complement: Absolute & Relative

- Given the Universe U, and A,B ⊂ U.
- The (absolute) complement of A is A=U\A
- The (relative) complement of A in B is B\A





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Proving Set Equivalences

- Recall that to prove such identity, we must show that:
  1. The left-hand side is a subset of the right-hand side
  2. The right-hand side is a subset of the left-hand side
  3. Then conclude that the two sides are thus equal
- The book proves several of the standard set identities
- We will give a couple of different examples here

Proving Set Equivalences: Example A (1)

- Let
  - $A=\{x|x \text{ is even}\}$
  - $B=\{x|x \text{ is a multiple of } 3\}$
  - $C=\{x|x \text{ is a multiple of } 6\}$
- Show that $A \cap B = C$

Proving Set Equivalences: Example A (2)

- **A∩B ⊆ C**: ∀ x ∈ A∩B
  - ⇒ x is a multiple of 2 and x is a multiple of 3
  - ⇒ we can write x=2x3 k for some integer k
  - ⇒ x=6k for some integer k ⇒ x is a multiple of 6
  - ⇒ x ∈ C
- **C ⊆A∩B**: ∀ x∈ C
  - ⇒ x is a multiple of 6 ⇒ x=6k for some integer k
  - ⇒ x=2(3k)=3(2k) ⇒ x is a multiple of 2 and of 3
  - ⇒ x ∈ A∩B

Proving Set Equivalences: Example B (1)

- An alternative prove is to use membership tables where an entry is
  - 1 if a chosen (but fixed) element is in the set
  - 0 otherwise
- Example: Show that

$$\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$$

Proving Set Equivalences: Example B (2)

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
0	0	0	0	**1**	1	1	1	**1**
0	0	1	0	**1**	1	1	0	**1**
0	1	0	0	**1**	1	0	1	**1**
0	1	1	0	**1**	1	0	0	**1**
1	0	0	0	**1**	0	1	1	**1**
1	0	1	0	**1**	0	1	0	**1**
1	1	0	0	**1**	0	0	1	**1**
1	1	1	1	**0**	0	0	0	**0**

- 1 under a set indicates that an element is in the set
- If the columns are equivalent, we can conclude that indeed the two sets are equal

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Generalizing Set Operations: Union and Intersection

- In the previous example, we showed De Morgan's Law generalized to unions involving 3 sets
- In fact, De Morgan's Laws hold for any finite set of sets
- Moreover, we can generalize set operations union and intersection in a straightforward manner to any finite number of sets

Generalized Union

- **Definition**: The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

LaTeX: $\Bigcup_{i=1}^{n}A_i=A_1\cup A_2 \cup\ldots\cup A_n$

Generalized Intersection

- **Definition**: The intersection of a collection of sets is the set that contains those elements that are members of every set in the collection

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

LaTex: $\Bigcap_{i=1}^{n}A_i=A_1\cap A_2 \cap\ldots\cap A_n$

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Computer Representation of Sets (1)

- There really aren't ways to represent infinite sets by a computer since a computer has a finite amount of memory
- If we assume that the universal set U is finite, then we can easily and effectively represent sets by bit vectors
- Specifically, we force an ordering on the objects, say:

$$U=\{a_1, a_2,\ldots,a_n\}$$

- For a set $A\subseteq U$, a bit vector can be defined as, for $i=1,2,\ldots,n$
  - $b_i=0$ if $a_i \notin A$
  - $b_i=1$ if $a_i \in A$

Computer Representation of Sets (2)

- Examples
  - Let U={0,1,2,3,4,5,6,7} and A={0,1,6,7}
  - The bit vector representing A is: 1100 0011
  - How is the empty set represented?
  - How is U represented?
- Set operations become trivial when sets are represented by bit vectors
  - Union is obtained by making the bit-wise OR
  - Intersection is obtained by making the bit-wise AND

Computer Representation of Sets (3)

- Let U={0,1,2,3,4,5,6,7}, A={0,1,6,7}, B={0,4,5}
- What is the bit-vector representation of B?
- Compute, bit-wise, the bit-vector representation of $A \cap B$
- Compute, bit-wise, the bit-vector representation of $A \cup B$
- What sets do these bit vectors represent?

Programming Question

- Using bit vector, we can represent sets of cardinality equal to the size of the vector
- What if we want to represent an arbitrary sized set in a computer (i.e., that we do not know a priori the size of the set)?
- What data structure could we use?

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Introduction

- You have already encountered function
  - $f(x,y) = x+y$
  - $f(x) = x$
  - $f(x) = \sin(x)$
- Here we will study functions defined on discrete domains and ranges
- We will generalize functions to mappings
- We may not always be able to write function in a 'neat way' as above

Definition: Function

- **Definition**: A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.
- We write *f*(a)=b if b is the unique element of B assigned by the function f to the element a $\in$ A.
- If *f* is a function from A to B, we write

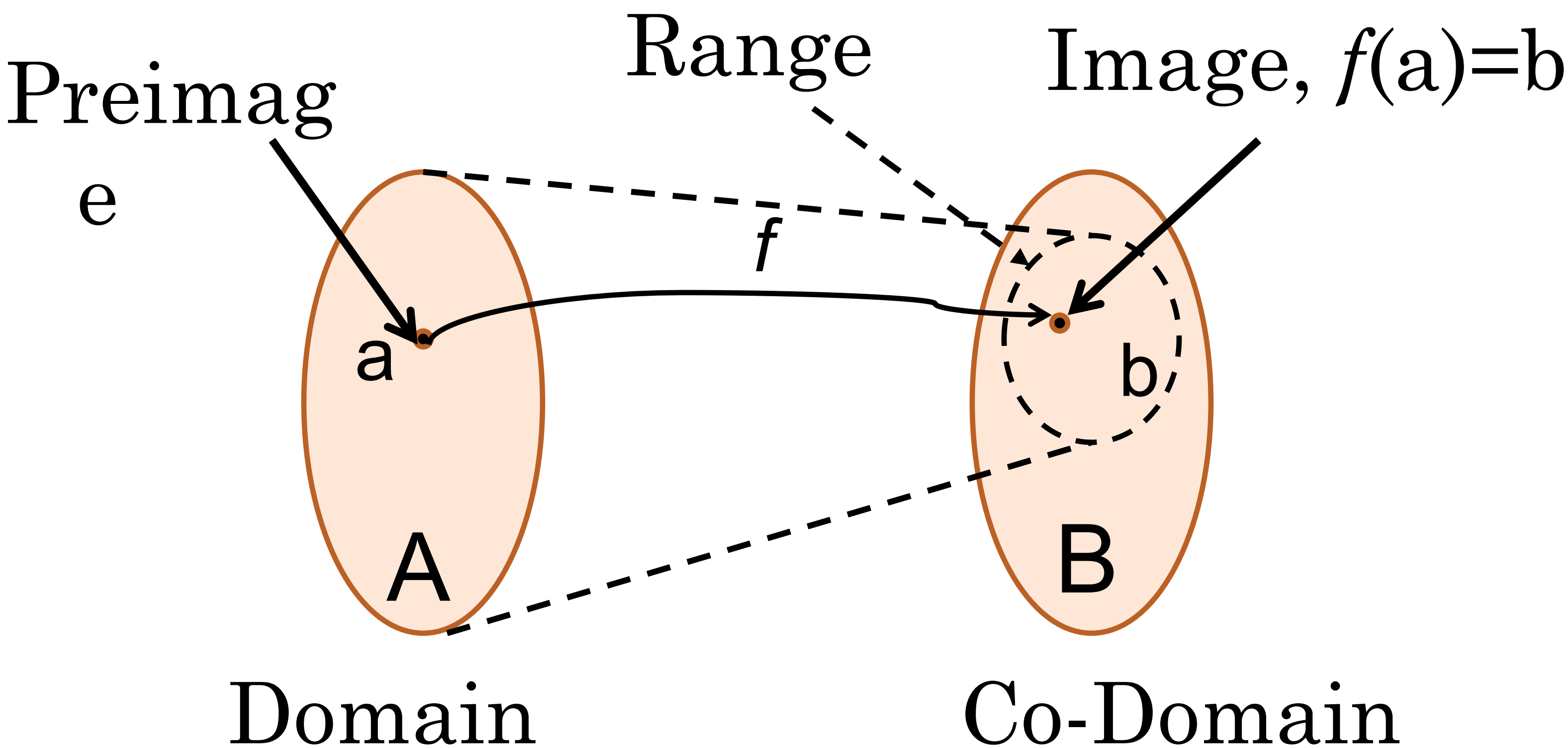
  $$f: A \rightarrow B$$

  This can be read as '*f* maps A to B'
- Note the subtlety
  - Each and every element of A has a <u>single</u> mapping
  - Each element of B may be mapped to by <u>several</u> elements in A or <u>not at all</u>

Terminology

- Let f: A $\rightarrow$ B and f(a)=b. Then we use the following terminology:
  - A is the domain of f, denoted dom(f)
  - B is the co-domain of f
  - b is the image of a
  - a is the preimage (antecedent) of b
  - The range of f is the set of all images of elements of A, denoted rng(f)

Function: Visualization



A function, f: A $\rightarrow$ B

More Definitions (1)

- **Definition**: Let f_1 and f_2 be two functions from a set A to R. Then f_1+f_2 and f_1f_2 are also function from A to R defined by:
  - $(f_1+f_2)(x) = f_1(x) + f_2(x)$
  - $f_1f_2(x)= f_1(x)f_2(x)$
- Example: Let $f_1(x)=x^4+2x^2+1$ and $f_2(x)=2-x^2$
  - $(f_1+f_2)(x) = x^4+2x^2+1+2-x^2 = x^4+x^2+3$
  - $f_1f_2(x) = (x^4+2x^2+1)(2-x^2)= -x^6+3x^2+2$

More Definitions (2)

- **Definition**: Let f: A$\rightarrow$ B and S $\subseteq$A. The image of the set S is the subset of B that consists of all the images of the elements of S. We denote the image of S by f(S), so that

$$f(S)=\{ f(s) \mid \forall\ s \in S \}$$

- Note there that the image of S is a set and not an element.

Image of a set: Example

- Let:
  - $A = \{a_1,a_2,a_3,a_4,a_5\}$
  - $B = \{b_1,b_2,b_3,b_4,b_5\}$
  - $f=\{(a_1,b_2), (a_2,b_3), (a_3,b_3), (a_4,b_1), (a_5,b_4)\}$
  - $S=\{a_1,a_3\}$
- Draw a diagram for *f*
- What is the:
  - Domain, co-domain, range of *f*?
  - Image of S, *f*(S)?

More Definitions (3)

- **Definition**: A function *f* whose domain and codomain are subsets of the set of real numbers (R) is called
  - strictly increasing if $f(x)<f(y)$ whenever $x<y$ and x and y are in the domain of *f*.
  - strictly decreasing if $f(x)>f(y)$ whenever $x<y$ and x and y are in the domain of *f*.
- A function that is increasing or decreasing is said to be monotonic

Next class

- Topic: Functions
- Pre-class reading: Chap 2.3