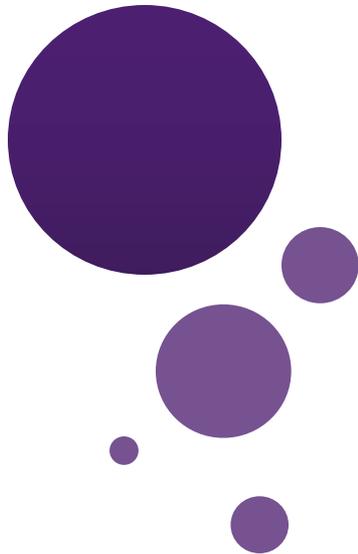




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Lecture 8: Functions

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Outline

- Properties
- Inverse functions (examples)
- Operators
- Important functions

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- **Properties**
- Inverse functions (examples)
- Operators
- Important functions

Definition: Injection

- **Definition:** A function f is said to be one-to-one or injective (or an injection) if
$$\forall x \text{ and } y \text{ in in the domain of } f, f(x)=f(y) \Rightarrow x=y$$
- Intuitively, an injection simply means that each element in the range has **at most** one preimage (antecedent)
- It is useful to think of the contrapositive of this definition

$$x \neq y \Rightarrow f(x) \neq f(y)$$

Definition: Surjection

- **Definition:** A function $f: A \rightarrow B$ is called onto or surjective (or an surjection) if

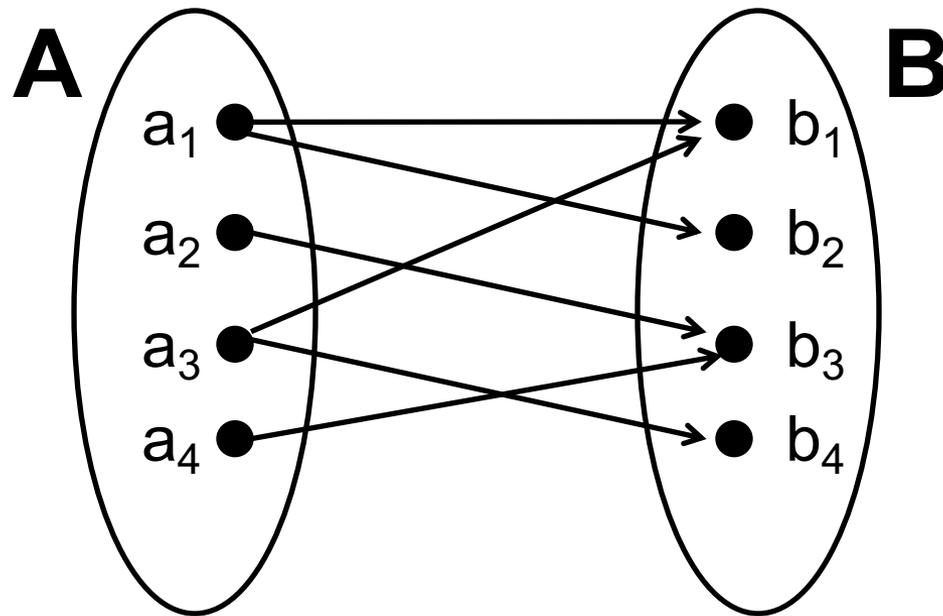
$$\forall b \in B, \exists a \in A \text{ with } f(a) = b$$

- Intuitively, a surjection means that every element in the codomain is mapped into (i.e., it is an image, has an antecedent)
- Thus, the range is the same as the codomain

Definition: Bijection

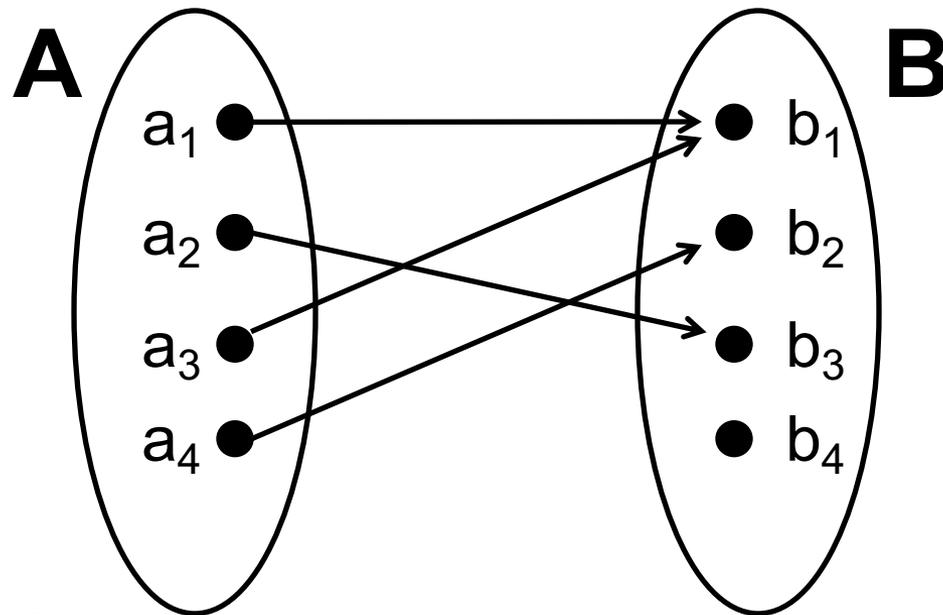
- **Definition:** A function f is a one-to-one correspondence (or a bijection), if it is both one-to-one (injective) and onto (surjective)
- One-to-one correspondences are important because they endow a function with an inverse.
- They also allow us to have a concept of cardinality for infinite sets
- Let's look at a few examples to develop a feel for these definitions...

Functions: Example 1



- Is this a function? Why?
- No, because each of a_1, a_2 has two images

Functions: Example 2

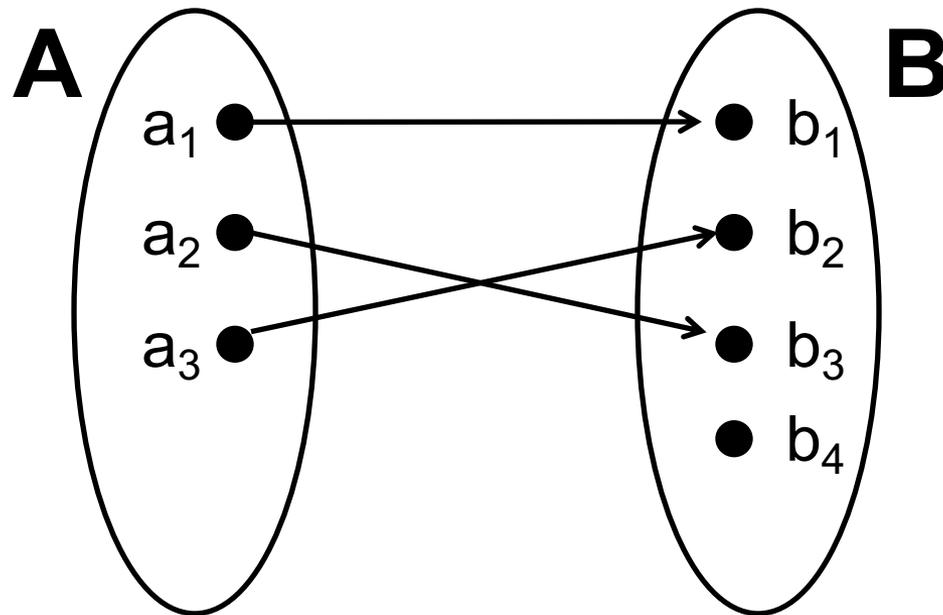


- Is this a function
 - One-to-one (injective)? Why?
 - Onto (surjective)? Why?

No, b_1 has 2 preimages

No, b_4 has no preimage

Functions: Example 3

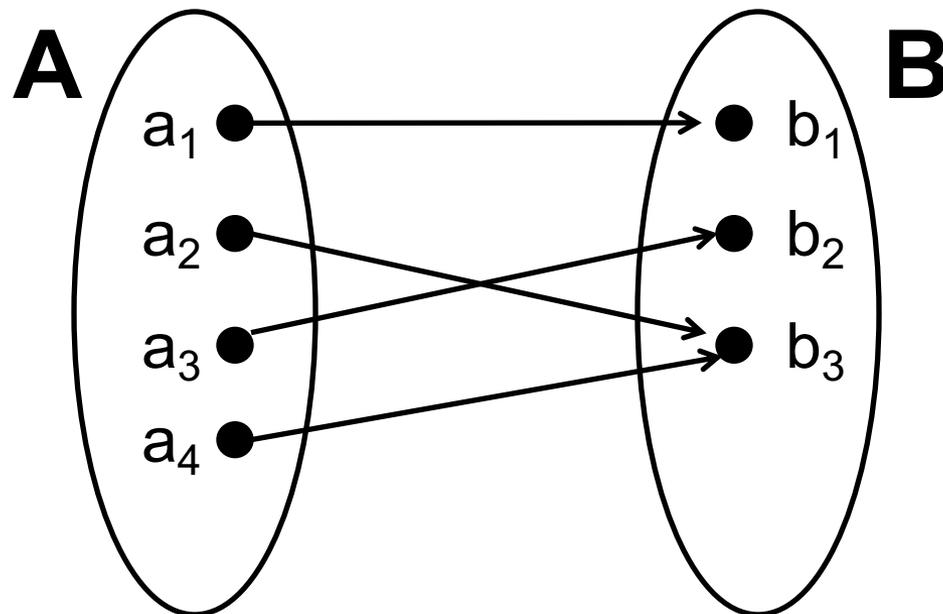


- Is this a function
 - One-to-one (injective)? Why?
 - Onto (surjective)? Why?

Yes, no b_i has 2 preimages

No, b_4 has no preimage

Functions: Example 4

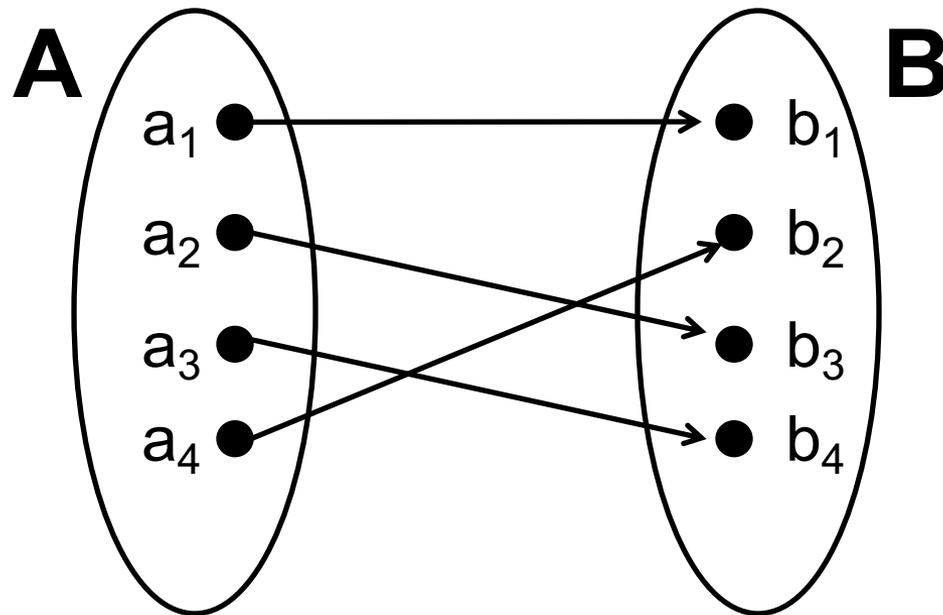


- Is this a function
 - One-to-one (injective)? Why?
 - Onto (surjective)? Why?

No, b_3 has 2 preimages

Yes, every b_i has a preimage

Functions: Example 5



- Is this a function
 - One-to-one (injective)?
 - Onto (surjective)?

Thus, it is a bijection or a one-to-one correspondence

Exercise 1

- Let $f:Z \rightarrow Z$ be defined by

$$f(x)=2x-3$$

- What is the domain, codomain, range of f ?
- Is f one-to-one (injective)?
- Is f onto (surjective)?
- Clearly, $\text{dom}(f)=Z$. To see what the range is, note that:

$$b \in \text{rng}(f) \Leftrightarrow b=2a-3, \text{ with } a \in Z$$

$$\Leftrightarrow b=2(a-2)+1$$

$$\Leftrightarrow b \text{ is odd}$$

Exercise 1 (cont'd)

- Thus, the range is the set of all odd integers
- Since the range and the codomain are different (i.e., $\text{rng}(f) \neq Z$), we can conclude that f is not onto (surjective)
- However, f is one-to-one injective. Using simple algebra, we have:

$$f(x_1) = f(x_2) \Rightarrow 2x_1 - 3 = 2x_2 - 3 \Rightarrow x_1 = x_2 \quad \text{QED}$$

Exercise 2

- Let f be as before

$$f(x) = 2x - 3$$

but now we define $f: N \rightarrow N$

- What is the domain and range of f ?
- Is f onto (surjective)?
- Is f one-to-one (injective)?
- By changing the domain and codomain of f , f is not even a function anymore. Indeed, $f(1) = 2 \cdot 1 - 3 = -1 \notin N$

Exercise 3

- Let $f:Z \rightarrow Z$ be defined by

$$f(x) = x^2 - 5x + 5$$

- Is this function
 - One-to-one?
 - Onto?

Exercise 3: Answer

- It is not one-to-one (injective)

$$\begin{aligned}f(x_1)=f(x_2) &\Rightarrow x_1^2-5x_1+5=x_2^2-5x_2+5 \Rightarrow x_1^2-5x_1=x_2^2-5x_2 \\ &\Rightarrow x_1^2-x_2^2=5x_1-5x_2 \Rightarrow (x_1-x_2)(x_1+x_2)=5(x_1-x_2) \\ &\Rightarrow (x_1+x_2)=5\end{aligned}$$

Many $x_1, x_2 \in \mathbb{Z}$ satisfy this equality. There are thus an infinite number of solutions. In particular, $f(2)=f(3)=-1$

- It is also not onto (surjective).

The function is a parabola with a global minimum at $(5/2, -5/4)$. Therefore, the function fails to map to any integer less than -1

- What would happen if we changed the domain/codomain?

Exercise 4

- Let $f:Z \rightarrow Z$ be defined by

$$f(x) = 2x^2 + 7x$$

- Is this function
 - One-to-one (injective)?
 - Onto (surjective)?
- Again, this is a parabola, it cannot be onto (where is the global minimum?)

Exercise 4: Answer

- However, it is one-to-one! Indeed:

$$\begin{aligned}f(x_1) = f(x_2) &\Rightarrow 2x_1^2 + 7x_1 = 2x_2^2 + 7x_2 \Rightarrow 2x_1^2 - 2x_2^2 = 7x_2 - 7x_1 \\ &\Rightarrow 2(x_1 - x_2)(x_1 + x_2) = 7(x_2 - x_1) \Rightarrow 2(x_1 + x_2) = -7 \Rightarrow (x_1 \\ &+ x_2) = -7/2 \\ &\Rightarrow (x_1 + x_2) = -7/2\end{aligned}$$

But $-7/2 \notin \mathbb{Z}$. Therefore it must be the case that $x_1 = x_2$.

It follows that f is a one-to-one function.

QED

Exercise 5

- Let $f:Z \rightarrow Z$ be defined by

$$f(x) = 3x^3 - x$$

- Is this function
 - One-to-one (injective)?
 - Onto (surjective)?

Exercise 5: f is one-to-one

- To check if f is one-to-one, again we suppose that for $x_1, x_2 \in \mathbb{Z}$ we have $f(x_1) = f(x_2)$

$$f(x_1) = f(x_2) \Rightarrow 3x_1^3 - x_1 = 3x_2^3 - x_2$$

$$\Rightarrow 3x_1^3 - 3x_2^3 = x_1 - x_2$$

$$\Rightarrow 3(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = (x_1 - x_2)$$

$$\Rightarrow (x_1^2 + x_1x_2 + x_2^2) = 1/3$$

which is impossible because $x_1, x_2 \in \mathbb{Z}$

thus, f is one-to-one

Exercise 5: f is not onto

- Consider the counter example $f(a)=1$
- If this were true, we would have
$$3a^3 - a = 1 \Rightarrow a(3a^2 - 1) = 1$$
 where a and $(3a^2 - 1) \in \mathbb{Z}$
- The only time we can have the product of two **integers** equal to 1 is when they are both equal to 1 or -1
- Neither 1 nor -1 satisfy the above equality
 - Thus, we have identified $1 \in \mathbb{Z}$ that does not have an antecedent and f is not onto (surjective)

Outline

- Properties
- **Inverse functions (examples)**
- Operators
- Important functions

Inverse Functions (1)

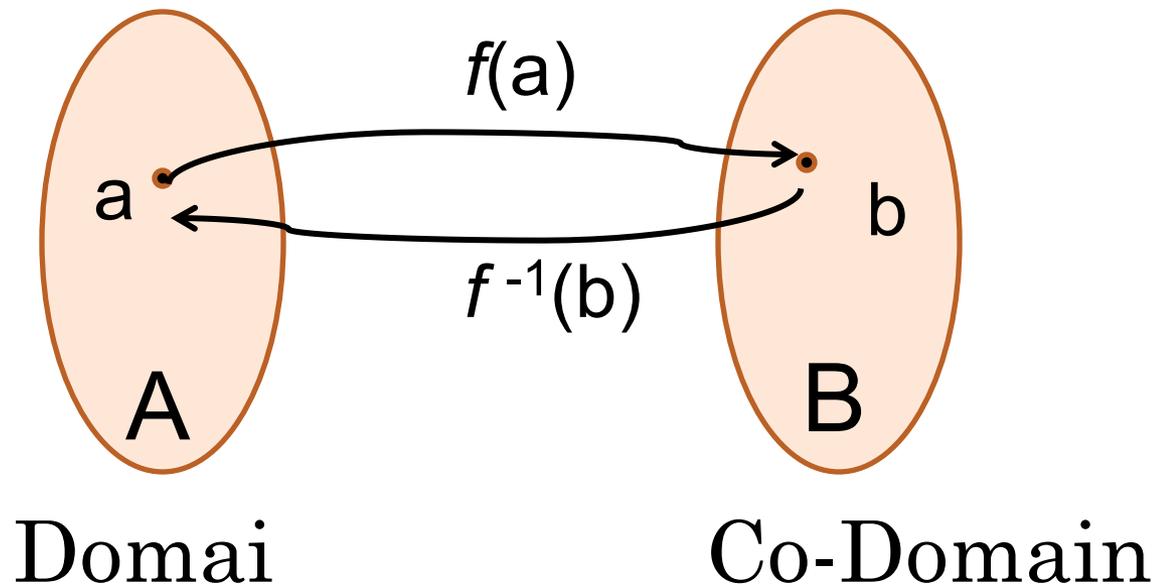
- **Definition:** Let $f: A \rightarrow B$ be a bijection. The inverse function of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$
- The inverse function is denoted f^{-1} .
- When f is a bijection, its inverse exists and

$$f(a) = b \iff f^{-1}(b) = a$$

Inverse Functions (2)

- Note that by definition, a function can have an inverse if and only if it is a bijection. Thus, we say that a bijection is invertible
- Why must a function be bijective to have an inverse?
 - Consider the case where f is not one-to-one (not injective). This means that some element $b \in B$ has more than one antecedent in A , say a_1 and a_2 . How can we define an inverse? Does $f^{-1}(b) = a_1$ or a_2 ?
 - Consider the case where f is not onto (not surjective). This means that there is some element $b \in B$ that does not have any preimage $a \in A$. What is then $f^{-1}(b)$?

Inverse Functions: Representation



η A function and its inverse

Inverse Functions: Example 1

- Let $f:R\rightarrow R$ be defined by

$$f(x) = 2x - 3$$

- What is f^{-1} ?

1. We must verify that f is invertible, that is, is a bijection. We prove that is one-to-one (injective) and onto (surjective). It is.
2. To find the inverse, we use the substitution
 - Let $f^{-1}(y)=x$
 - And $y=2x-3$, which we solve for x . Clearly, $x= (y+3)/2$
 - So, $f^{-1}(y)= (y+3)/2$

Inverse Functions: Example 2

- Let $f(x)=x^2$. What is f^{-1} ?
- No domain/codomain has been specified.
- Say $f:R\rightarrow R$
 - Is f a bijection? Does its inverse exist?
 - Answer: No
- Say we specify that $f: A \rightarrow B$ where
$$A=\{x\in R \mid x\leq 0\} \text{ and } B=\{y\in R \mid y\geq 0\}$$
 - Is f a bijection? Does its inverse exist?
 - Answer: Yes, the function becomes a bijection and thus, has an inverse

Inverse Functions: Example 2 (cont')

- To find the inverse, we let
 - $f^{-1}(y)=x$
 - $y=x^2$, which we solve for x
- Solving for x , we get $x=\pm\sqrt{y}$, but which one is it?
- Since $\text{dom}(f)$ is all nonpositive and $\text{rng}(f)$ is nonnegative, thus x must be nonpositive and
$$f^{-1}(y) = -\sqrt{y}$$
- From this, we see that the domains/codomains are just as important to a function as the definition of the function itself

Inverse Functions: Example 3

- Let $f(x)=2^x$
 - What should the domain/codomain be for this function to be a bijection?
 - What is the inverse?
- The function should be $f:R\rightarrow R^+$
- Let $f^{-1}(y)=x$ and $y=2^x$, solving for x we get $x=\log_2(y)$. Thus, $f^{-1}(y)=\log_2(y)$
- What happens when we include 0 in the codomain?
- What happens when restrict either sets to Z ?

Outline

- Properties
- Inverse functions (examples)
- **Operators**
- Important functions

Function Composition (1)

- The value of functions can be used as the input to other functions
- **Definition:** Let $g:A\rightarrow B$ and $f:B\rightarrow C$. The composition of the functions f and g is

$$(f \circ g)(x) = f(g(x))$$

- $f \circ g$ is read as ' f circle g ', or ' f composed with g ', ' f following g ', or just ' f of g '
- In LaTeX: `\circ`

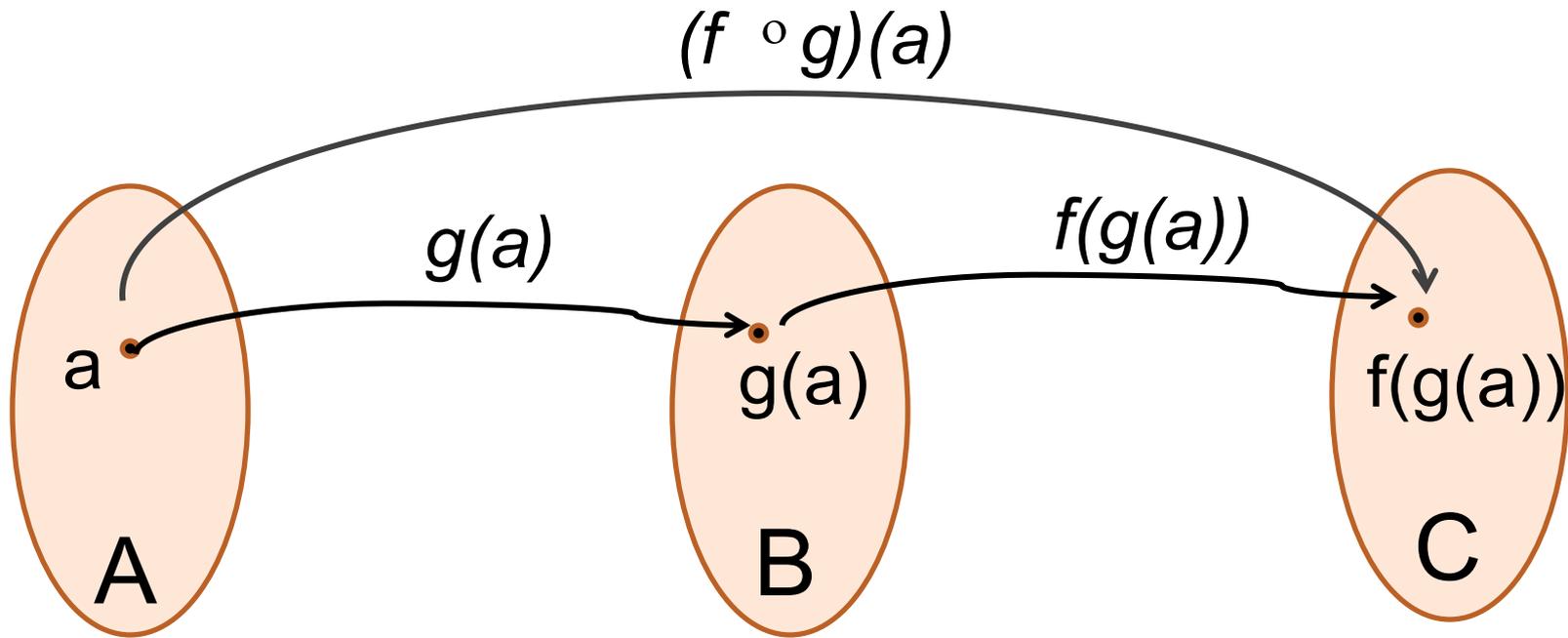
Function Composition (2)

- Because $(f \circ g)(x) = f(g(x))$, the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f

$$f \circ g \text{ is defined } \Leftrightarrow \text{rng}(g) \subseteq \text{dom}(f)$$

- The order in which you apply a function matters: you go from the inner most to the outer most
- It follows that $f \circ g$ is in general not the same as $g \circ f$

Composition: Graphical Representation



The composition of two functions

Composition: Example 1

- Let f, g be two functions on $R \rightarrow R$ defined by

$$f(x) = 2x - 3$$

$$g(x) = x^2 + 1$$

- What are $f \circ g$ and $g \circ f$?
- We note that
 - f is bijective, thus $\text{dom}(f) = \text{rng}(f) = \text{codomain}(f) = R$
 - For g , $\text{dom}(g) = R$ but $\text{rng}(g) = \{x \in R \mid x \geq 1\} \subseteq R^+$
 - Since $\text{rng}(g) = \{x \in R \mid x \geq 1\} \subseteq R^+ \subseteq \text{dom}(f) = R$, $f \circ g$ is defined
 - Since $\text{rng}(f) = R \subseteq \text{dom}(g) = R$, $g \circ f$ is defined

Composition: Example 1 (cont')

- Given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$
- $(f \circ g)(x) = f(g(x)) = f(x^2+1) = 2(x^2+1)-3$
 $= 2x^2 - 1$
- $(g \circ f)(x) = g(f(x)) = g(2x-3) = (2x-3)^2 + 1$
 $= 4x^2 - 12x + 10$

Function Equality

- Although it is intuitive, we formally define what it means for two functions to be equal
- **Lemma:** Two functions f and g are equal if and only
 - $\text{dom}(f) = \text{dom}(g)$
 - $\forall a \in \text{dom}(f) (f(a) = g(a))$

Associativity

- The composition of function is not commutative ($f \circ g \neq g \circ f$), it is associative
- **Lemma:** The composition of functions is an associative operation, that is

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Outline

- Properties
- Inverse functions (examples)
- Operators
- **Important functions**

Important Functions: Identity

- **Definition:** The identity function on a set A is the function

$$\iota: A \rightarrow A$$

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defined by $\iota(a) = a$ for all $a \in A$.

- One can view the identity function as a composition of a function and its inverse:

$$\iota(a) = (f \circ f^{-1})(a) = (f^{-1} \circ f)(a)$$

- Moreover, the composition of any function f with the identity function is itself f :

$$(f \circ \iota)(a) = (\iota \circ f)(a) = f(a)$$

Inverses and Identity

- The identity function, along with the composition operation, gives us another characterization of inverses when a function has an inverse
- **Theorem:** The functions $f: A \rightarrow B$ and $g: B \rightarrow A$ are inverses if and only if

$$(g \circ f) = \iota_A \text{ and } (f \circ g) = \iota_B$$

where the ι_A and ι_B are the identity functions on sets A and B . That is,

$$\forall a \in A, b \in B ((g(f(a))) = a) \wedge (f(g(b)) = b))$$

Important Functions: Absolute Value

- **Definition:** The absolute value function, denoted $|x|$, $f: \mathbb{R} \rightarrow \{y \in \mathbb{R} \mid y \geq 0\}$. Its value is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

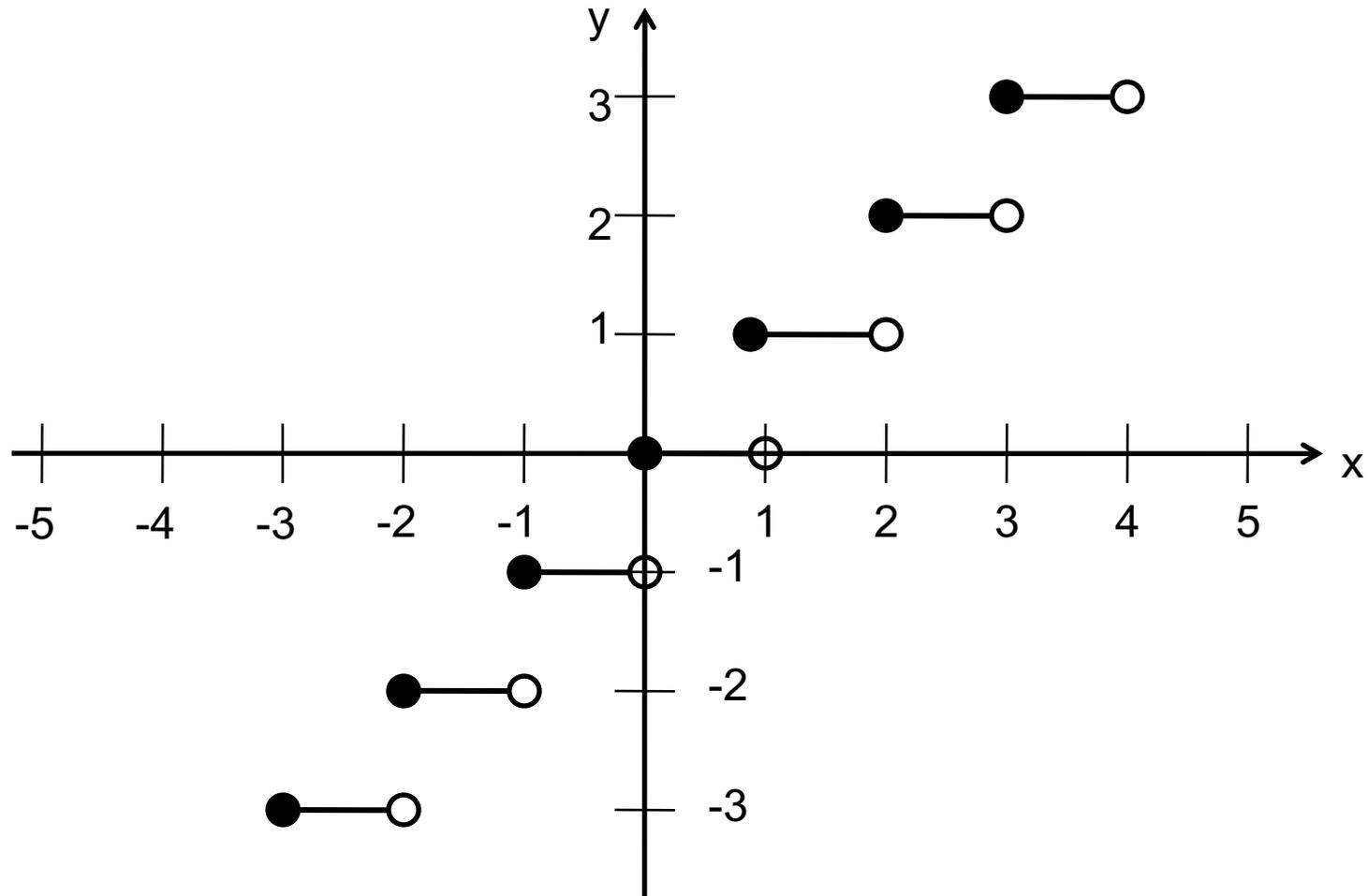
Important Functions: Floor & Ceiling

- **Definitions:**

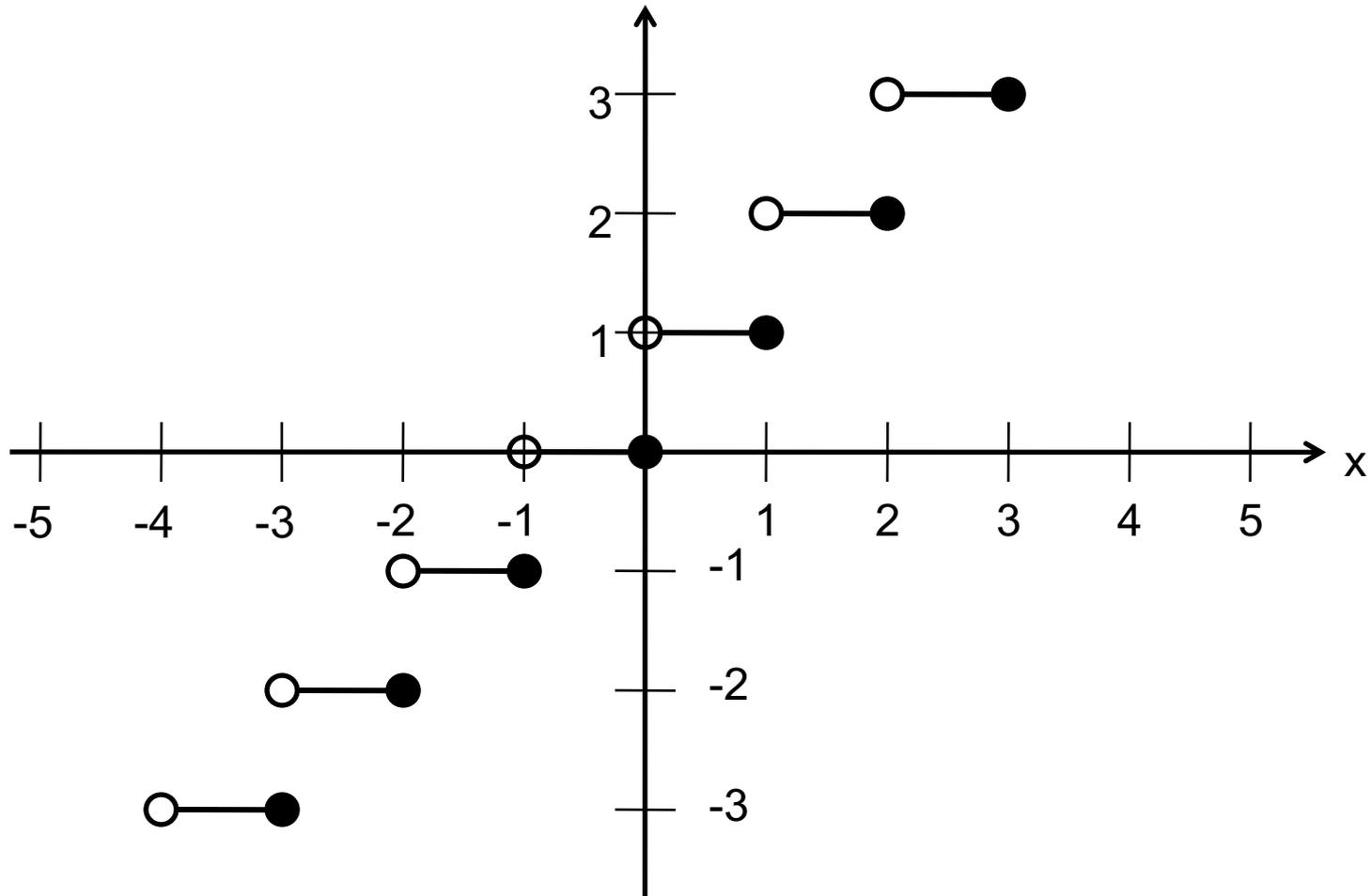
- The floor function, denoted $\lfloor x \rfloor$, is a function $R \rightarrow Z$. Its values is the largest integer that is less than or equal to x
- The ceiling function, denoted $\lceil x \rceil$, is a function $R \rightarrow Z$. Its values is the smallest integer that is greater than or equal to x

- In LaTeX: $\$\\lceil$, $\$\\rceil$, $\$\\rfloor$, $\$\\lfloor$

Important Functions: Floor



Important Functions: Ceiling



Important Function: Factorial

- The factorial function gives us the number of permutations (that is, uniquely ordered arrangements) of a collection of n objects
- **Definition:** The factorial function, denoted $n!$, is a function $N \rightarrow N^+$. Its value is the product of the n positive integers

$$n! = \prod_{i=1}^{i=n} i = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$$

Factorial Function & Stirling's Approximation

- The factorial function is defined on a discrete domain
- In many applications, it is useful a continuous version of the function (say if we want to differentiate it)
- To this end, we have the Stirling's formula

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Summary

- Properties
 - One-to-one (injective), onto (surjective), one-to-one correspondence (bijective)
- Inverse functions (examples)
- Operators
 - Composition, Equality
- Important functions
 - identity, absolute value, floor, ceiling, factorial

Next class

- Topic: Sequences
- Pre-class reading: Chap 2.4

