Outline

- Sequences
- Progressions: Special sequences
- Summations

Lecture 9: Sequences and Summations (1)

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- **Sequences**
- Progressions: Special sequences
- Summations

Sequence

A sequence is an *ordered* list of elements.

- A sequence is often given as
  - $a_1, a_2, \ldots, a_n, \ldots$
  - a_n is a term in the sequence.

- A sequence is actually a function *f* from a subset of **Z** to a set S
  - Usually from the positive or non-negative integers
  - a_n is the image of n: $f(n) = a_n$

Examples of Sequence

- $a_n = 3n$
  - The terms in the sequence are $a_1, a_2, a_3, \ldots$
  - The sequence $\{a_n\}$ is $\{ 3, 6, 9, 12, \ldots \}$

- $b_n = 2^n$
  - The terms in the sequence are $b_1, b_2, b_3, \ldots$
  - The sequence $\{b_n\}$ is $\{ 2, 4, 8, 16, 32, \ldots \}$

Examples of Sequence

- The difference is in how they grow
- Arithmetic sequences increase by a constant *amount*
  - $a_n = 3n$
  - The sequence is { 3, 6, 9, 12, … }
  - Each number is 3 more than the last
  - Of the form: $f(x) = dx + a$
- Geometric sequences increase by a constant *factor*
  - $b_n = 2^n$
  - The sequence is { 2, 4, 8, 16, 32, … }
  - Each number is twice the previous
  - Of the form: $f(x) = ar^x$

Examples of Sequence

- Sequences can be neither geometric or arithmetic
  - $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
    - Alternative, $F(n) = F(n-1) + F(n-2)$
  - Each term is the sum of the previous two terms
  - Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
  - This is the Fibonacci sequence
  - Full formula: $F(n) = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5} \cdot 2^n}$

Examples of Sequence

Not all sequences are arithmetic or geometric sequences.

An example is Fibonacci sequence

- $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
  - Alternative, $F(n) = F(n\text{-}1) + F(n\text{-}2)$
- Each term is the sum of the previous two terms
- Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
- This is the Fibonacci sequence

More on Fibonacci Sequence

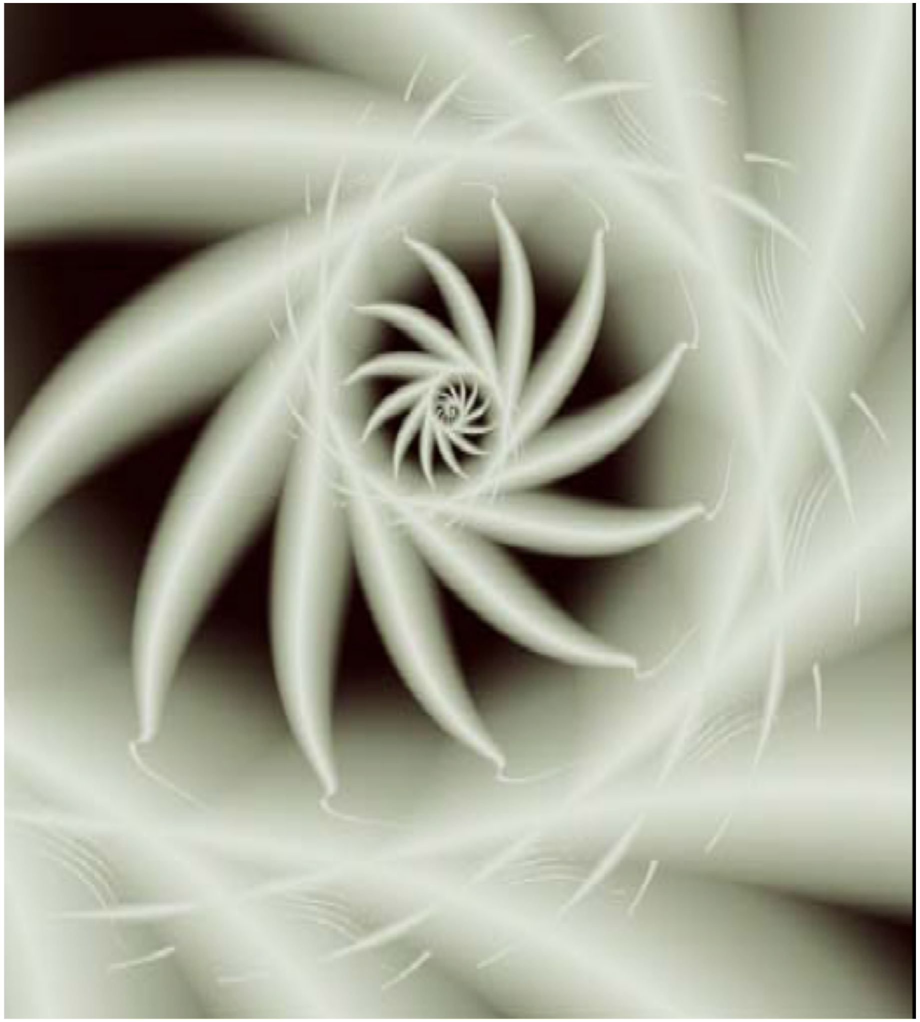
- As the terms increase, the ratio between successive terms approaches 1.618

$$\lim_{n\to\infty}\frac{F(n+1)}{F(n)}=\phi=\frac{\sqrt{5}+1}{2}=1.618933989$$

- This is called the “golden ratio”
  - Ratio of human leg length to arm length
  - Ratio of successive layers in a conch shell

- Reference: http://en.wikipedia.org/wiki/Golden_ratio

Examples of Golden Ratio



Sequence Formula

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …

- The sequence alternates 1's and 0's, increasing the number of 1's and 0's each time

b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …

- This sequence increases by one, but repeats all even numbers once

c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …

- The non-0 numbers are a geometric sequence (2^n) interspersed with zeros

d) 3, 6, 12, 24, 48, 96, 192, …

- Each term is twice the previous: geometric progression
- $a_n = 3*2^{n-1}$

Sequence Formula

e) 15, 8, 1, -6, -13, -20, -27, …

- Each term is 7 less than the previous term
- $a_n = 22 - 7n$

f) 3, 5, 8, 12, 17, 23, 30, 38, 47, …

- The difference between successive terms increases by on each time
- $a_1 = 3$, $a_n = a_{n-1} + n$
- $a_n = n(n+1)/2 + 2$

g) 2, 16, 54, 128, 250, 432, 686, …

- Each term is twice the cube of n
- $a_n = 2 * n^3$

h) 2, 3, 7, 25, 121, 721, 5041, 40321

- Each successive term is about n times the previous
- $a_n = n! + 1$

Some useful sequences

- $n^2 = 1, 4, 9, 16, 25, 36, \ldots$
- $n^3 = 1, 8, 27, 64, 125, 216, \ldots$
- $n^4 = 1, 16, 81, 256, 625, 1296, \ldots$
- $2^n = 2, 4, 8, 16, 32, 64, \ldots$
- $3^n = 3, 9, 27, 81, 243, 729, \ldots$
- $n! = 1, 2, 6, 24, 120, 720, \ldots$

Sequences

- **Definition**: A sequence is a function from a subset of integers to a set S. We use the notation(s):

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

- Each a_n is called the n^{th} term of the sequence
- We rely on the context to distinguish between a sequence and a set, although they are distinct structures

Sequences: Example 2

- The sequence: $\{h_n\}_{n=1}^{\infty}$ = 1/n
  is known as the **harmonic** sequence
- The sequence is simply:

  1, 1/2, 1/3, 1/4, 1/5, …

- This sequence is particularly interesting because its summation is divergent:

  $\sum_{n=1}^{\infty}$ (1/n) = ∞

Sequences: Example 1

- Consider the sequence

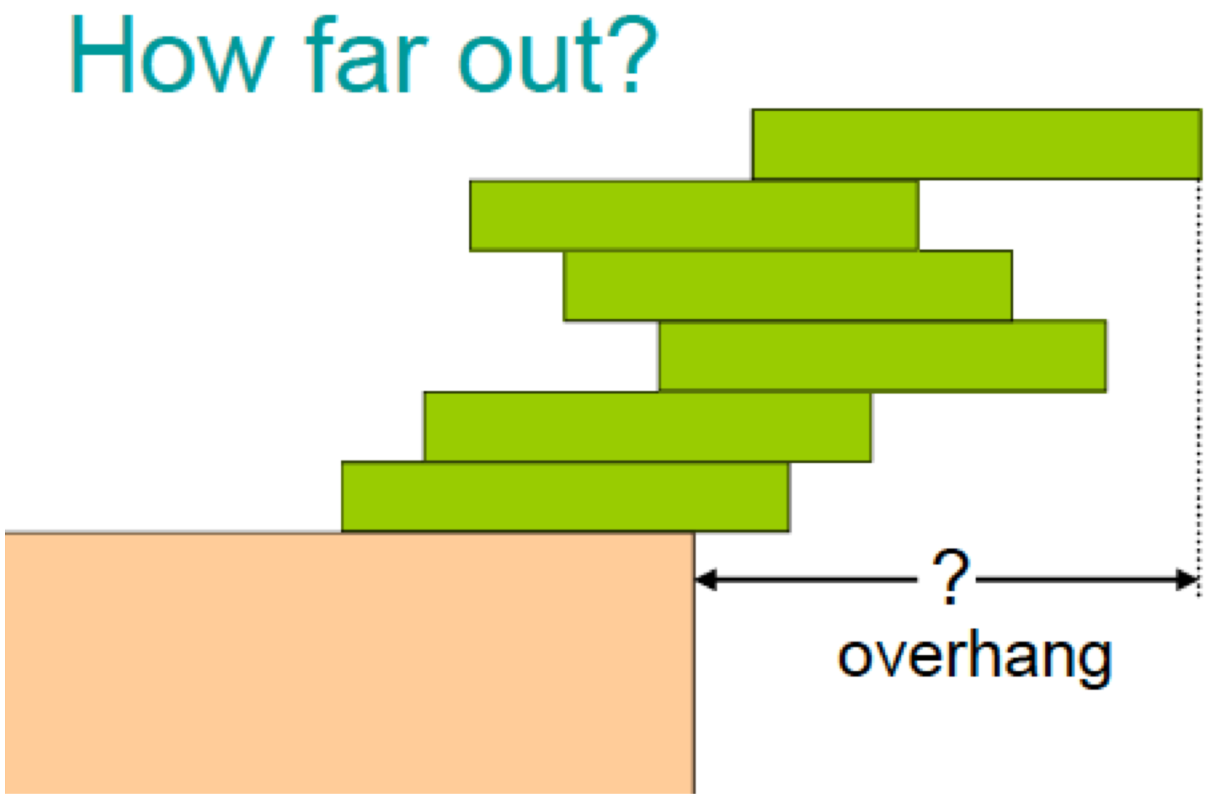
$$\{(1 + 1/n)^n\}_{n=1}^{\infty}$$

- The terms of the sequence are:

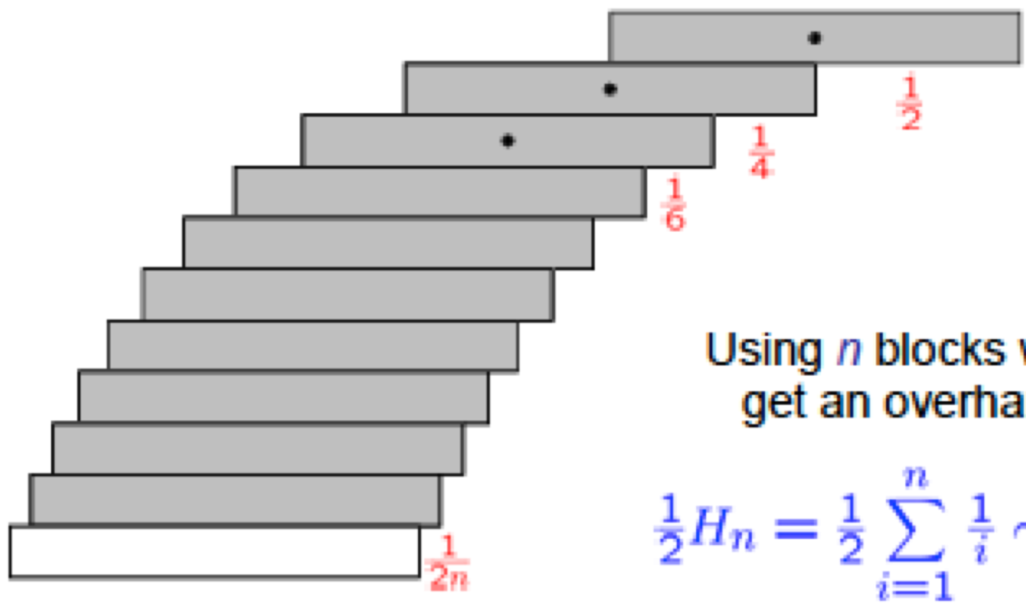
$$a_1 = (1 + 1/1)^1 = 2.00000$$
$$a_2 = (1 + 1/2)^2 = 2.25000$$
$$a_3 = (1 + 1/3)^3 = 2.37037$$
$$a_4 = (1 + 1/4)^4 = 2.44140$$
$$a_5 = (1 + 1/5)^5 = 2.48832$$

- What is this sequence?
- The sequence corresponds to $\lim_{n\to\infty}\{(1 + 1/n)^n\}_{n=1}^{\infty} = e = 2.71828..$

Book stacking example



Book stacking example



Using n blocks we can get an overhang of

$$\frac{1}{2}H_n = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{i} \sim \frac{1}{2}\ln n$$

Harmonic Stacks

Outline

- Sequences
- **Progressions: Special sequences**
- Summations

Progressions: Geometric

- **Definition**: A geometric progression is a sequence of the form

$$a, aq, aq^2, aq^3, \ldots, aq^n, \ldots$$

Where:

  - $a \in R$ is called the initial term
  - $q \in R$ is called the common ratio

- A geometric progression is a discrete analogue of the exponential function

$$f(x) = aq^x$$

Geometric Progressions: Examples

- A common geometric progression in Computer Science is:

  $$\{a_n\}= 1/2^n$$

  with a=1 and q=1/2
- Give the initial term and the common ratio of
  - $\{b_n\}$ with $b_n= (-1)^n$
  - $\{c_n\}$ with $c_n= 2(5)^n$
  - $\{d_n\}$ with $d_n= 6(1/3)^n$

Progressions: Arithmetic

- **Definition**: An <u>arithmetric progression</u> is a sequence of the form

  $$a, a+d, a+2d, a+3d, \ldots, a+nd, \ldots$$

  Where:
  - $a \in R$ is called the <u>initial term</u>
  - $d \in R$ is called the <u>common difference</u>
- An arithmetic progression is a <u>discrete</u> analogue of the linear function

  $$f(x) = dx+a$$

Arithmetic Progressions: Examples

- Give the initial term and the common difference of
  - $\{s_n\}$ with $s_n = -1 + 4n$
  - $\{t_n\}$ with $s_n = 7 – 3n$

Outline

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- **Summations**

Summations

- You should be by now familiar with the summation notation:

  $$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

  Here
  - i is the index of the summation
  - m is the lower limit
  - n is the upper limit
- Often times, it is useful to change the lower/upper limits, which can be done in a straightforward manner (although we must be very careful):

  $$\sum_{i=1}^{n} a_i = \sum_{i=0}^{n-1} a_{i+1}$$

Infinite Series: Geometric Series

- In fact, we can generalize that fact as follows
- **Lemma**: A geometric series converges if and only if the absolute value of the common ratio is less than 1

$$\Sigma_{i=0}^{n} (aq^i) = \begin{cases} (aq^{n+1}-a)/(q-1) & \text{if } q \neq 1 \\ (n+1)a & \text{if } q = 1 \end{cases}$$

Sum of geometric series

$$G_n ::= 1 + x + x^2 + \cdots + x^{n-1} + x^n$$

What is the closed form expression of G_n?

$$G_n = \sum_{i=0}^{n} x^i$$

$$G_n ::= 1 + x + x^2 + \cdots + x^{n-1} + x^n$$

$$xG_n = x + x^2 + x^3 + \cdots + x^n + x^{n+1}$$

$$G_n - xG_n = 1 - x^{n+1}$$

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

Sum of arithmetic series

Given n numbers, a_1, a_2, …, a_n with common difference d, i.e. $a_{i+1} - a_i = d$.

What is a simple closed form expression of the sum?

$$S_n = \sum_{i=1}^{n} a_i$$

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots\ldots + (a_1 + (n-2)d) + (a_1 + (n-1)d)$$

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-2)d) + (a_n - (n-1)d)$$

Adding the equations together gives:

$$2S_n = n(a_1 + a_n)$$

Rearranging and remembering that $a_n = a_1 + (n - 1)d$, we get:

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n[2a_1 + (n-1)d]}{2}$$

Next class

- Topic: Sequences and Summation (2)
- Pre-class reading: Chap 2.4